FREIBURGER INTERVALL-BERICHTE

81/10

R. Angelov; S. Markov:

Extended Segment Analysis

J. Garloff:

Die Auflösung linearer Toeplitz - Intervallgleichungssysteme

M. Raith:

Inklusions-isotone, zentrierte Kugel-Erweiterungen

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EXTENDED SEGMENT ANALYSIS

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Abstract.

This paper can be considered as a continuation of [4], where the elements of a theory based both on extended interval analysis [1,2,3] and on segment analysis [5,6] are presented. The theory thus obtained leads to some interesting results on interval functions which can be considered as generalizations of classical theorems for real functions. Some applications of these results to problems of existence and uniqueness of solutions of first order differential equations are given.

1. The interval space $\langle I(\overline{R}),+,-,\cdot\rangle$

Let R be the set of reals and let $\overline{R} = R \cup \{-\infty\} \cup \{\infty\}$. We shall denote the set of all closed intervals on $\overline{R}$ by $I(\overline{R})$, and the set of all compact intervals on R by I(R). Intervals with coinciding end-points are also admitted so that $R \subset I(R)$ and $\overline{R} \subset I(\overline{R})$. If $a \in R$ then [a,a] is a point interval and $a = [a,a] \in I(R) \subset I(\overline{R})$. The void interval $\emptyset$ is also an element of I(R) and $I(\overline{R})$.

Partial orderings in $I(\overline{R})$. Let $A = [\underline{a},\overline{a}]$ and $B = [\underline{b},\overline{b}]$ be elements of $I(\overline{R})$. We say that A is less than or equal to B and write $A \leqq B$ if $\underline{a} \leqq \underline{b}$ and $\overline{a} \leqq \overline{b}$. We thus introduce a partial ordering in $I(\overline{R})$. Similarly the relations $<, \geqq, >$ are introduced.

Another partial ordering in $I(\overline{R})$ is based on the relation inclusion; in this case the intervals are considered as sub-sets of $\overline{R}$. More precisely, A contains B (symbolically $A \supset B$), if $\underline{a} \leqq \underline{b}$ and $\overline{a} \geqq \overline{b}$.

Set-theoretic operations. The joint of two intervals $A = [\underline{a},\overline{a}]$, $B = [\underline{b},\overline{b}]$ is called the interval $A \vee B = [\min\{\underline{a},\underline{b}\}, \max\{\overline{a},\overline{b}\}]$, that is $A \vee B$ is the minimum interval containing both A and B. The joint of two intervals is also an interval and therefore the concept of joint is different from the concept of union of intervals in the sense of set theory.

By intersection of two intervals A and B we mean their intersection in set-theoretical sense. The intersection of two elements of $I(\bar{R})$ is also an element of $I(\bar{R})$ and more precisely

$$A \wedge B = \begin{cases} [\max\{\underline{a},\underline{b}\}, \min\{\bar{a},\bar{b}\}], & \text{if } A \wedge B \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Arithmetic operations in $I(\bar{R})$. Let $A = [\underline{a},\bar{a}]$ and $B = [\underline{b},\bar{b}]$ be elements of $I(R)$. The interval $A + B = [\underline{a}+\underline{b}, \bar{a}+\bar{b}]$ is the sum of the intervals A and B. The interval $A - B = [(\underline{a}-\underline{b}) \vee (\bar{a}-\bar{b})]$ is the difference of A and B. The number $w(A) = \bar{a}-\underline{a}$ is called the width of A. We thus have

$$A - B = \begin{cases} [\underline{a}-\underline{b}, \bar{a}-\bar{b}], & \text{if } w(A) \geq w(B), \\ [\bar{a}-\bar{b}, \underline{a}-\underline{b}], & \text{if } w(A) < w(B). \end{cases}$$

Let $\alpha \in R$. The interval $\alpha A = \{[\alpha\underline{a},\alpha\bar{a}],\ \alpha \geq 0;\ [\alpha\bar{a},\alpha\underline{a}],\ \alpha < 0\}$ is called the product of α and A. The following properties hold true for these three operations:

1. $(A+B)+C = A+(B+C)$;
2. $A+B = B+A$;

3. For every $A \in I(R)$ we have $0 \cdot A = [0,0] = 0$;

4. If $\alpha\beta \geqq 0$, then $(\alpha+\beta)A = \alpha A+\beta A$ and $(\alpha-\beta)A = \alpha A-\beta A$;

5. $\alpha(A+B) = \alpha A+\alpha B$; $\alpha(A-B) = \alpha A-\alpha B$;

6. $(\alpha\beta)A = \alpha(\beta A)$;

7. $1 \cdot A = A$;

8. For every two elements A and C belonging to $I(R)$ such that $w(A) \leqq w(C)$ there exists $B \in I(R)$, so that $A+B = C$. For every two elements belonging to $I(R)$ such that $w(A) \geqq w(C)$ there exists an element $B \in I(R)$, so that $A-B = C$.

The space $\langle I(R),+,-,\cdot\rangle$ is called an interval space [2]. We shall extend this space to the space $\langle I(\overline{R}),+,-,\cdot\rangle$ by extending the arithmetic operations for infinite intervals. To this end it is sufficient to set: $(+\infty)+\alpha = +\infty$ for $\alpha \in R \cup (+\infty)$; $(+\infty)+(-\infty) = [-\infty,+\infty]$; $(-\infty)+\alpha = -\infty$ for $\alpha \in R \cup (-\infty)$; $(-\infty)+(+\infty) = [-\infty,+\infty]$; $(+\infty)-\alpha = +\infty$ for $\alpha \in R \cup (-\infty)$; $(+\infty)-(+\infty) = 0$; $(-\infty)-\alpha = -\infty$ for $\alpha \in R \cup (+\infty)$; $(-\infty)-(-\infty) = 0$; $0\cdot(\pm\infty) = [-\infty,+\infty]$; $\alpha\cdot(\pm\infty) = \pm\infty$ for $\alpha \in R \setminus (0)$.

For example, if $\alpha,\beta \in R$, then we have $[\alpha,+\infty]+[\beta,+\infty] = [\alpha+\beta,+\infty]$; $[\alpha,+\infty]-[-\infty,\beta] = [+\infty,+\infty] = +\infty$; $[\alpha,+\infty]-[\beta,+\infty] = [(\alpha-\beta) \vee 0]$; $0\cdot[\alpha,+\infty] = [-\infty,+\infty]$; $[-\infty,\alpha]+[+\infty,+\infty] = [-\infty,+\infty]$; $0\cdot[-\infty,\beta] = [-\infty,+\infty]$.

We note that $A-B \neq A+(-B)$, where $-B = (-1)\cdot B$. The equality $A-B = A+(-B)$ takes place exactly when $A \in R$ or $B \in R$. Similarly, $A+B \neq A-(-B)$ in general and $A+B = A-(-B)$ exactly when $A \in R$ or $B \in R$. For convenience, we use the following notations: $A \ominus B = A+(-B)$, $A \oplus B = A-(-B)$.

It is easily seen that $A \oplus B \subset A+B$ and $A-B \subset A \ominus B$. We also have $A \oplus B = B \oplus A$; $\alpha(A \oplus B) = \alpha A \oplus \alpha B$; $\alpha(A \ominus B) = \alpha A \ominus \alpha B$.

Let $A, B, C, D \in I(R)$. Denote $\mu_1 = \mu_1(A,B,C,D) = (w(A)-w(C))(w(B)-w(D))$, $\mu_2 = \mu_2(A,B,C,D) = (w(A)-w(B))(w(C)-w(D))$. Then:

9. $$(A+B) - (C+D) = \begin{cases} (A-C) + (B-D), & \text{if } \mu_1 \geqq 0, \\ (A-C) \oplus (B-D), & \text{if } \mu_1 < 0; \end{cases}$$

10. $$(A-B) + (C-D) = \begin{cases} (A+C) - (B+D), & \text{if } \mu_2 \geqq 0, \\ (A \oplus C) - (B \oplus D), & \text{if } \mu_2 < 0,\ \mu_1 \geqq 0, \\ (A \oplus C) \ominus (B \oplus D), & \text{if } \mu_2 < 0,\ \mu_1 < 0; \end{cases}$$

11. $$(A-B) - (C-D) = \begin{cases} (A-C) - (B-D), & \text{if } \mu_2 \geqq 0,\ \mu_1 \geqq 0, \\ (A-C) \ominus (B-D), & \text{if } \mu_2 \geqq 0,\ \mu_1 < 0, \\ (A \ominus C) - (B \ominus D), & \text{if } \mu_2 < 0; \end{cases}$$

12. $$(A-B) \oplus (C-D) = \begin{cases} (A \oplus C) - (B \oplus D), & \text{if } \mu_2 \geqq 0,\ \mu_1 \geqq 0, \\ (A \oplus C) \ominus (B \oplus D), & \text{if } \mu_2 \geqq 0,\ \mu_1 < 0, \\ (A+C) - (B+D), & \text{if } \mu_2 < 0; \end{cases}$$

13. $$(A-C) \oplus (B-D) \subset (A+B) - (C+D) \subset (A-C) + (B-D).$$

Interval norm in $I(\overline{R})$. Consider the function $\|\cdot\|: I(\overline{R}) \to [0,+\infty]$ defined for every interval $A = [\underline{a},\overline{a}]$ by $\|A\| = \max\{|\underline{a}|,|\overline{a}|\}$. This function is an interval norm in the following sense:

1. $\|A\| \geq 0$ and $\|A\| = 0$, iff $A = [0,0] = 0$,
2. $\|\alpha A\| = |\alpha| \, \|A\|$,
3. $\|A+B\| \leq \|A\| + \|B\|$,

for every $A,B \in I(R)$ and $\alpha \in R$.

Moreover we have $\|A-B\| \leq \|A\|+\|B\|$.

2. Interval functions. S-limit and S-continuity.

By an interval function we mean a function G, which takes values in the interval space $I(\overline{R})$. We shall assume that G is defined in some sub-set Ω of the metric space L.

Let $z \in \Omega$. The endpoints of the interval G(z) will be denoted by $\underline{g}(z)$ and $\overline{g}(z)$. Thus we determine two real functions $\underline{g}$ and $\overline{g}$ on Ω such that $G(z) = [\underline{g}(z),\overline{g}(z)] = \underline{g}(z) \vee \overline{g}(z)$. Conversely, if φ and ψ are two arbitrary real functions defined on Ω, then we can define an interval function by means of $G(z) = \varphi(z) \vee \psi(z)$.

Limit and continuity of interval functions. Let $G(z) = [\underline{g}(z),\overline{g}(z)]$ be an interval function defined on Ω. We say that z_o is a limit point of Ω, if every neighbourhood of z_o contains points of Ω.

Let z_o be a limit point of Ω. We say that G(z) has a limit when $z \to z_o$ if the limits $\lim_{z\to z_o} \underline{g}(z)$ and $\lim_{z\to z_o} \overline{g}(z)$ exist. We call the interval $\lim_{z\to z_o} G(z) = [\lim_{z\to z_o}\underline{g}(z),\lim_{z\to z_o}\overline{g}(z)]$ the limit of G when $z \to z_o$.

For arbitrary interval functions F,G which have a limit when $z \to z_o$ we have $\lim_{z\to z_o}(F(z) * G(z)) = (\lim_{z\to z_o}F(z)) * (\lim_{z\to z_o}G(z))$, where $*$ is an arbitrary arithmetic operation in $\langle I(\overline{R}),+,-,\cdot\rangle$.

We say that G is continuous at $z_o \in \Omega$, if it has a limit when $z \to z_o$ and $G(z_o) = \lim_{z\to z_o} G(z)$, that is $\underline{g}(z_o) = \lim_{z\to z_o} \underline{g}(z)$, $\overline{g}(z_o) = \lim_{z\to z_o} \overline{g}(z)$.

<u>S-limit of interval functions</u>. Let $G(z) = [\underline{g}(z), \overline{g}(z)]$ be defined on $\Omega \subset L$ and z_o be a limit point of Ω.

<u>Definition 1</u>. We call the interval $[\underline{\lim}_{z\to z_o} \underline{g}(z), \overline{\lim}_{z\to z_o} \overline{g}(z)]$ *the S-limit of G when* $z \to z_o$ *and denote*

$$S\text{-}\lim_{z\to z_o} G(z) = [\underline{\lim}_{z\to z_o} \underline{g}(z), \overline{\lim}_{z\to z_o} \overline{g}(z)].$$

Every function has a S-limit when $z \to z_o$ if z_o is a limit point. For example consider the situation when G is defined in the set of naturals, that is $G(n)=G^{(n)}$, $n=1,2,\ldots,$ is a sequence of intervals. In this case the S-limit of $\{G^{(n)}\}_{n=1}^{\infty}$ when $n \to \infty$ is

$$S\text{-}\lim_{n\to\infty} G^{(n)} = [\underline{\lim}_{n\to\infty} \underline{g}^{(n)}, \overline{\lim}_{n\to\infty} \overline{g}^{(n)}].$$

The following definitions of S-limit are equivalent:

<u>Definition 2</u>. $S\text{-}\lim_{z\to z_o} G(z) = \wedge_{\delta>0} \vee_{0<|z-z_o|<\delta,\, z\in\Omega} G(z)$.

<u>Definition 3</u>. $S\text{-}\lim_{z\to z_o} G(z) = \lim_{\delta\to 0} \vee_{0<|z-z_o|<\delta,\, z\in\Omega} G(z)$.

The functions $\underline{\varphi}(\delta)$ and $\overline{\varphi}(\delta)$, defined by $[\underline{\varphi}(\delta),\overline{\varphi}(\delta)] = \phi(\delta) = \vee_{0<|z-z_o|<\delta, z\in\Omega} G(z)$ are monotonic and therefore the limits $\lim_{\delta\to 0}\underline{\varphi}(\delta)$ and $\lim_{\delta\to 0}\overline{\varphi}(\delta)$ exist, so that $\lim_{\delta\to 0} \phi(\delta)$ exists as well.

The equivalence of the above three definitions is easy to be noticed. Indeed, we have $[\underline{\lim}_{z\to z_o}\underline{g}(z),\ \underline{\lim}_{z\to z_o}\overline{g}(z)]$
$= [\lim_{\delta\to 0}\inf_{0<|z-z_o|<\delta, z\in\Omega}\underline{g}(z),\ \lim_{\delta\to 0}\sup_{0<|z-z_o|<\delta, z\in\Omega}\overline{g}(z)]$
$= \lim_{\delta\to 0}[\inf_{0<|z-z_o|<\delta, z\in\Omega}\underline{g}(z), \sup_{0<|z-z_o|<\delta, z\in\Omega}\overline{g}(z)]$
$= \lim_{\delta\to 0}\vee_{0<|z-z_o|<\delta, z\in\Omega}G(z)$.

On the other hand $\wedge_{\delta>0}\phi(\delta) \subset \phi(\delta)$ so that $\wedge_{\delta>0}\phi(\delta) \subset \lim_{\delta\to 0}\phi(\delta)$. We also have $\phi(\delta_1) \supset \phi(\delta)$ for $\delta_1 > \delta$ which implies $\phi(\delta_1) \supset \lim_{\delta\to 0}\phi(\delta)$ and $\wedge_{\delta>0}\phi(\delta) \supset \lim_{\delta\to 0}\phi(\delta)$. We thus obtain $\wedge_{\delta>0}\phi(\delta) = \lim_{\delta\to 0}\phi(\delta)$.

We shall note some elementary properties of the S-limit.

1. Let $A \in I(R)$ and $\omega \in \{\underline{\leq},\underline{\geq},\supset,\subset\}$. If the relation $G(z)\omega A$ holds true for every z from some neighbourhood of z_o, then we have $(S\text{-}\lim_{z\to z_o} G(z))\omega A$.

2. Let $A = S\text{-}\lim_{z\to z_o} G(z)$. Then for every $\varepsilon>0$ there exists $\delta>0$ such that $A+E \supset G(z)$ for $0 < |z-z_o| < \delta$, $z \in \Omega$, where $E = [-\varepsilon,\varepsilon]$.

3. If $\lim_{z\to z_o} G(z)$ exists, then $S\text{-}\lim_{z\to z_o} G(z) = \lim_{z\to z_o} G(z)$.

Theorems for S-limits.

Theorem 1. If the interval functions $F(z)$ and $G(z)$ are defined on $\Omega \subset R$ and z_o is a limit point of Ω, then

$$S\text{-}lim_{z \to z_o} (F(z) \vee G(z)) = S\text{-}lim_{z \to z_o} F(z) \vee S\text{-}lim_{z \to z_o} G(z).$$

Proof. Denote $H(z) = F(z) \vee G(z) = [\min\{\underline{f}(z), \underline{g}(z)\}, \max\{\overline{f}(z), \overline{g}(z)\}] = [\underline{h}(z), \overline{h}(z)]$. We have $S\text{-}\lim_{z \to z_o} H(z) = [\underline{\lim}_{z \to z_o} \underline{h}(z), \overline{\lim}_{z \to z_o} \overline{h}(z)] = [\underline{\lim}_{z \to z_o} \min\{\underline{f}(z), \underline{g}(z)\}, \overline{\lim}_{z \to z_o} \max\{\overline{f}(z), \overline{g}(z)\}] = [\min\{\underline{\lim}_{z \to z_o} \underline{f}(z), \underline{\lim}_{z \to z_o} \underline{g}(z)\}, \max\{\overline{\lim}_{z \to z_o} \overline{f}(z), \overline{\lim}_{z \to z_o} \overline{g}(z)\}] = S\text{-}\lim_{z \to z_o} F(z) \vee S\text{-}\lim_{z \to z_o} G(z)$.

This theorem reveals the natural relation between the operation "v" and the concept of S-limit. A similar relation does not exist for the other operations in $I(\overline{R})$.

Theorem 2. Let F and G be interval functions defined on $\Omega \subset L$ and z_o be a limit point of Ω. Then we have

$$S\text{-}lim_{z \to z_o} (F(z) * G(z)) \subset S\text{-}lim_{z \to z_o} F(z) * S\text{-}lim_{z \to z_o} G(z)$$

for $* \in \{+, \ominus\}$,

$$S\text{-}lim_{z \to z_o} (F(z) * G(z)) \supset S\text{-}lim_{z \to z_o} F(z) * S\text{-}lim_{z \to z_o} G(z)$$

for $* \in \{-, \oplus\}$.

Proof. Let $* = +$. We have $\underline{\lim}_{z \to z_o} (\underline{f}(z) + \underline{g}(z)) \geq \underline{\lim}_{z \to z_o} \underline{f}(z) + \underline{\lim}_{z \to z_o} \underline{g}(z)$; $\overline{\lim}_{z \to z_o} (\overline{f}(z) + \overline{g}(z)) \leq \overline{\lim}_{z \to z_o} \overline{f}(z) + \overline{\lim}_{z \to z_o} \overline{g}(z)$

and therefore $S\text{-}\lim_{z\to z_o}(F(z)+G(z)) =$
$[\underline{\lim}_{z\to z_o}(\underline{f}(z)+\underline{g}(z)),\overline{\lim}_{z\to z_o}(\overline{f}(z)+\overline{g}(z))] \subset [\underline{\lim}_{z\to z_o}\underline{f}(z)+$
$\underline{\lim}_{z\to z_o}\underline{g}(z),\overline{\lim}_{z\to z_o}\overline{f}(z)+\overline{\lim}_{z\to z_o}\overline{g}(z)] = [\underline{\lim}_{z\to z_o}\underline{f}(z),$
$\overline{\lim}_{z\to z_o}\overline{f}(z)] + [\underline{\lim}_{z\to z_o}\underline{g}(z),\overline{\lim}_{z\to z_o}\overline{g}(z)] =$
$S\text{-}\lim_{z\to z_o}F(z) + S\text{-}\lim_{z\to z_o}G(z)$.

Let $* = -$. It is easy to check that the following inequations hold true: $\underline{\lim}_{z\to z_o}(\underline{f}(z)-\underline{g}(z)) \leqq$
$\underline{\lim}_{z\to z_o}\underline{f}(z)-\underline{\lim}_{z\to z_o}\underline{g}(z) \leqq \overline{\lim}_{z\to z_o}(\underline{f}(z)-\underline{g}(z))$;
$\underline{\lim}_{z\to z_o}(\overline{f}(z)-\overline{g}(z)) \leqq \overline{\lim}_{z\to z_o}\overline{f}(z)-\overline{\lim}_{z\to z_o}\overline{g}(z) \leqq \overline{\lim}_{z\to z_o}$
$(\overline{f}(z)-\overline{g}(z))$.

Further, we have $S\text{-}\lim_{z\to z_o}F(z)- S\text{-}\lim_{z\to z_o}G(z) =$
$(\underline{\lim}_{z\to z_o}\underline{f}(z)-\underline{\lim}_{z\to z_o}\underline{g}(z)) \vee (\overline{\lim}_{z\to z_o}\overline{f}(z)-\overline{\lim}_{z\to z_o}\overline{g}(z)) \subset$
$[\underline{\lim}_{z\to z_o}(\underline{f}(z)-\underline{g}(z)),\overline{\lim}_{z\to z_o}(\underline{f}(z)-\underline{g}(z))] \vee [\underline{\lim}_{z\to z_o}(\overline{f}(z)-\overline{g}(z)),$
$\overline{\lim}_{z\to z_o}(\overline{f}(z)-\overline{g}(z))] = S\text{-}\lim_{z\to z_o}(\underline{f}(z)-\underline{g}(z)) \vee$
$S\text{-}\lim_{z\to z_o}(\overline{f}(z)-\overline{g}(z)) = S\text{-}\lim_{z\to z_o}((\underline{f}(z)-\underline{g}(z)) \vee (\overline{f}(z)-\overline{g}(z))) =$
$S\text{-}\lim_{z\to z_o}([\underline{f}(z),\overline{f}(z)]-[\underline{g}(z),\overline{g}(z)]) = S\text{-}\lim_{z\to z_o}(F(z)-G(z))$.

For the operations $\oplus$ and $\ominus$ analogous arguments take place.

<u>*Theorem 3.*</u> *Let $F(z)$ and $G(z)$ be defined on $\Omega \subset L$ and z_o be a limit point of Ω. If $\lim_{z\to z_o}G(z)$ exists, then*

1. *$S\text{-}lim_{z\to z_o}(F(z) * G(z)) = S\text{-}lim_{z\to z_o} F(z) * S\text{-}lim_{z\to z_o} G(z)$ for $* \in \{+, \ominus\}$;*

2. *If $w(F(z)) \geq w(G(z))$ in some neighbourhood of z_o, then $S\text{-}lim_{z\to z_o}(F(z) * G(z)) = S\text{-}lim_{z\to z_o} F(z) * S\text{-}lim_{z\to z_o} G(z)$, $* \in \{-, \oplus\}$.*

<u>Proof</u>. By the above assumption we have $\text{S-lim}_{z\to z_o}(F(z)+G(z)) = [\underline{\lim}_{z\to z_o}\underline{f}(z)+\lim_{z\to z_o}\underline{g}(z), \overline{\lim}_{z\to z_o}\overline{f}(z)+\lim_{z\to z_o}\overline{g}(z)] = \text{S-lim}_{z\to z_o}F(z)+\lim_{z\to z_o}G(z)$.

Furthermore, $\text{S-lim}_{z\to z_o}(F(z)-G(z)) = \text{S-lim}_{z\to z_o}[(\underline{f}(z)-\underline{g}(z)), (\overline{f}(z)-\overline{g}(z))] = [\underline{\lim}_{z\to z_o}(\underline{f}(z)-\underline{g}(z)), \overline{\lim}_{z\to z_o}(\overline{f}(z)-\overline{g}(z))] = [\underline{\lim}_{z\to z_o}\underline{f}(z)-\lim_{z\to z_o}\underline{g}(z), \overline{\lim}_{z\to z_o}\overline{f}(z)-\lim_{z\to z_o}\overline{g}(z)] = \text{S-lim}_{z\to z_o}F(z)-\lim_{z\to z_o}G(z)$.

The arguments for the operations $\ominus$ and $\oplus$ are analogous.

<u>*Theorem 4*</u>. *Let $\alpha(z)$ and $G(z)$ be a real and an interval function respectively, defined on $\Omega \subset L$ and z_o be a limit point of Ω and the limit $lim_{z\to z_o}\alpha(z) = \alpha_o$ exist. Then $S\text{-}lim_{z\to z_o}(\alpha(z)G(z)) \subset \alpha_o S\text{-}lim_{z\to z_o}G(z)$. Moreover, if at least one of the following conditions holds true: a) $\alpha_o \neq 0$; b) $||S\text{-}lim_{z\to z_o}G(z)|| < +\infty$; then $S\text{-}lim_{z\to z_o}(\alpha(z)G(z)) = \alpha_o S\text{-}lim_{z\to z_o}G(z)$.*

Proof. Let $\alpha_o > 0$. Then $\alpha(z) > 0$ in some neighbourhood of z_o and thus $S\text{-}\lim_{z\to z_o}(\alpha(z)G(z)) = [\underline{\lim}_{z\to z_o}\alpha(z)\underline{g}(z), \overline{\lim}_{z\to z_o}\alpha(z)\overline{g}(z)] = [\alpha_o\underline{\lim}_{z\to z_o}\underline{g}(z), \alpha_o\overline{\lim}_{z\to z_o}\overline{g}(z)] = \alpha_o S\text{-}\lim_{z\to z_o}G(z)$.

For $\alpha_o < 0$ analogous considerations hold true.

Let $\alpha_o = 0$ and $\|S\text{-}\lim_{z\to z_o}G(z)\| < +\infty$. Then $S\text{-}\lim_{z\to z_o}(\alpha(z)G(z)) = [\lim_{z\to z_o}\alpha(z)\underline{g}(z) \vee \lim_{z\to z_o}\alpha(z)\overline{g}(z)] = [0,0] = 0 = \alpha_o S\text{-}\lim_{z\to z_o}G(z)$.

If $\alpha_o = 0$ and $S\text{-}\lim_{z\to z_o}G(z)$ is an infinite interval, then it follows that $\alpha_o S\text{-}\lim_{z\to z_o}G(z) = [-\infty,+\infty]$ and the inclusion is obvious.

3. S-continuity of interval functions

Let the interval function G be defined on $\Omega \subset L$ and $z_o \in \Omega$ be a limit point of Ω and r be the metric in L.

Definition. We say that G is S-continuous at z_o if $S\text{-}\lim_{z\to z_o} G(z) \subset G(z_o)$ and that G is S-continuous on Ω, if it is S-continuous at every limit point of Ω.

Complete graph of the interval functions. Let $E \subset \Omega$. Consider the functions:

$$I(E,G,\delta;z) = \inf\{y: y \in G(t),\ r(z,t) \leq \delta,\ t \in E\},$$

$$S(E,G,\delta;z) = \sup\{y: y \in G(t),\ r(z,t) \leq \delta,\ t \in E\},$$

$$I(E,G;z) = \lim_{\delta\to o} I(E,G,\delta;z),$$

$$S(E,G;z) = \lim_{\delta\to o} S(E,G,\delta;z).$$

Definition. The complete graph of G on E is the interval function $F(E,G;z) = [I(E,G;z),\ S(E,G;z)]$ [5].

We see that $F(E,G;z)$ is defined for all points of $\bar{E}$: the closure of E with respect to the topology on L. We obviously have $G(z) \subset F(E,G;z)$ for $z \in E$. When $E = \Omega$ we shall sometimes write $F(\Omega,G;z) = F(G;z)$.

Theorem 5. The interval function G, defined on Ω is S-continuous on Ω exactly when $G(z) = F(G;z)$ for every $z \in \Omega$ [5].

Proof. Let $F(G;z) = G(z)$ for $z \in \Omega$ and let $z_o \in \Omega$ be a limit point of Ω. Then $S\text{-}\lim_{z\to z_o} G(z) = \lim_{\delta\to o} \vee_{o<|z-z_o|<\delta, z\in\Omega} G(z) \subset \lim_{\delta\to o}[I(G,\delta;z_o), S(G,\delta;z_o)] = F(G;z_o) = G(z_o)$.

Let now G be S-continuous on Ω and let $z_o \in \Omega$ be a limit point of Ω. We choose $\varepsilon > 0$. Since $S\text{-}\lim_{z\to z_o} G(z) \subset G(z_o)$, there exists $\delta > 0$, such that $G(z_o)+E \supset (\vee_{o<|z-z_o|\leq\delta, z\in\Omega} G(z))$ $\vee\, G(z_o) = \vee_{|z-z_o|\leq\delta, z\in\Omega} G(z) = [I(G,\delta;z_o), S(G,\delta;z_o)]$. Hence for $\delta \to 0$ we obtain $G(z_o)+E \supset F(G;z_o)$. Since $\varepsilon > 0$ is arbitrary and $G(z_o)$ is a closed interval, then $G(z_o) \supset F(G;z_o)$. But $G(z_o) \subset F(G;z_o)$ and therefore $G(z_o) = F(G;z_o)$.

In the situation when z_o is an isolated point of Ω the equality $G(z_o) = F(G;z_o)$ is obvious. Thus the theorem is proved.

We shall further denote the set of all S-continuous functions on Ω by F_Ω.

Theorem 6. Let $G(z) = [\underline{g}(z), \overline{g}(z)]$ be defined on Ω. Then $G \in F_\Omega$ exactly when $\underline{g}$ and $\overline{g}$ are respectively lower and upper semi-continuous on Ω.

Proof. If $\underline{g}$ and $\overline{g}$ are respectively lower and upper semi-continuous, then for every limit point $z_o \in \Omega$ we have $S\text{-}\lim_{z\to z_o} G(z) = [\underline{\lim}_{z\to z_o} \underline{g}(z), \overline{\lim}_{z\to z_o} \overline{g}(z)] \subset [\underline{g}(z_o), \overline{g}(z_o)] = G(z_o)$.

Conversely, if G is S-continuous, then $S\text{-}\lim_{z\to z_o} G(z) \subset G(z_o)$ implies $\underline{\lim}_{z\to z_o} \underline{g}(z) \geqq \underline{g}(z_o)$ and $\overline{\lim}_{z\to z_o} \overline{g}(z) \leqq \overline{g}(z_o)$, which means that $\underline{g}$ and $\overline{g}$ are respectively lower and upper semi-continuous.

Theorem 7. If Ω is a closed sub-set of L, then $G \in F_\Omega$ exactly when the graph of G is a closed set, that is the set $\{(z,y): z \in \Omega,\ y \in G(z)\}$ is closed in $L \times R$.

The verification of Theorem 7. is easy and we shall omit it.

Theorems for S-continuous functions. In what follows we shall consider some properties for the S-continuous functions, which are analogous to corresponding properties of the continuous real functions.

Theorem 8. If $F(z), G(z) \in F_\Omega$, then $H(z) = F(z) * G(z) \in F_\Omega$ for $* \in \{+, \ominus, \vee\}$.

<u>Proof</u>. Let $z_o \in \Omega$ be a limit point of Ω. Then from Theorem 1. and Theorem 2. it follows that
$S\text{-}\lim_{z\to z_o}(F(z) * G(z)) \subset S\text{-}\lim_{z\to z_o} F(z) * S\text{-}\lim_{z\to z_o} G(z) \subset$
$F(z_o) * G(z_o) = H(z_o)$.

An such assertion does not hold true for the operations - and $\oplus$. For example, let $\Omega = [-1,1]$ and

$$F(x) = \begin{cases} 0, & x\neq 0, \\ [0,1], & x=0; \end{cases} \qquad G(x) = \begin{cases} 1, & x\neq 0, \\ [0,1], & x=0. \end{cases}$$

Then $F(z) - G(z) = \{-1,\ x\neq 0;\ 0,\ x=0\}$ is not S-continuous.

<u>Theorem 9</u>. If $G \in F_\Omega$ and f is a continuous real function then $H(z) = G(z) * f(z)$, where $* \in \{-, \oplus, \cdot\}$, is S-continuous.

The proof follows directly from Theorem 3. and Theorem 4.

<u>Theorem 10</u>. If G is S-continuous on $\Omega \subset L_1$ and h is a continuous mapping from $D \subset L_1$ into Ω, where L_1 and L_2 are metric spaces, then the function $H(t) = G(h(t))$ is S-continuous in D.

<u>Proof</u>. Let $t_o \in D$ be a limit point of D. From the continuity of h it follows that $h(t) \to h(t_o)$ when $t \to t_o$. Then

$\text{S-lim}_{t\to t_o} H(t) = \text{S-lim}_{t\to t_o} G(h(t)) \subset \text{S-lim}_{z\to h(t_o)} G(z) \subset$ $G(h(t_o)) = H(t_o)$.

Theorem 11. Let Ω be a compact sub-set of L, $G \in F_\Omega$ and $G(z)$ be a finite interval for every $z \in \Omega$. Then the union of the values of G is a compact set.

Proof. $N = \{y: y \in G(z),\ z \in \Omega\}$. Let $\{y_n\}_{n=1}^{\infty} \subset N$ and $\{z_n\}_{n=1}^{\infty}$ be such that $y_n \in G(z_n)$. Since Ω is compact there exists a sub-sequence $\{z_{n_k}\}_{k=1}^{\infty}$ which converges to some $z_o \in \Omega$. Then $\text{S-lim}_{k\to\infty} G(z_{n_k}) \subset G(z_o)$ and consequently $\text{S-lim}_{k\to\infty} y_{n_k} \subset G(z_o)$. When $k > \nu$, $y_{n_k} \in G(z_o)+[-1,1]$ holds true. Since $G(z_o)+[-1,1]$ is a compact interval there exists a sub-sequence $\{y_{n_{k_s}}\}_{s=1}^{\infty}$ which converges to some $y_o \in G(z_o)+[-1,1]$. Using the fact that $\text{S-lim}_{s\to\infty} y_{n_{k_s}} \subset G(z_o)$ we obtain $y_o \in G(z_o) \subset N$, which proves the compactness of N.

Corollary 1. Let Ω be compact, $G \in F_\Omega$ and $\|G(z)\| < \infty$ for every $z \in \Omega$. Then there exist $\xi \in \Omega$ and $p \in R$ such that $p \in G(\xi)$ and $G(z) \leqq p$ for $z \in \Omega$.

In what follows we shall make use of the following relation $A \asymp B$ between two intervals $A, B \in I(\overline{R})$. By $A \asymp B$ we shall mean that either $A \subset B$ or $A \supset B$.

Theorem 12. *Let $\Omega = \Delta$ be a compact interval in R, $\Delta = [a,b]$, $G \in F_\Delta$ and $A \in I(R)$. If $A \subset G(a) \vee G(b)$, then there exists $\xi \in [a,b]$, such that $G(\xi) \curlyeqprec A$.*

Proof. In the situation when $A \subset G(a)$ or $A \subset G(b)$ the theorem is obvious. Let $A \not\subset G(a)$, $A \not\subset G(b)$. Without loss of generality we may assume that $G(a) \leqq A \leqq G(b)$. Denote $A = [c,d]$, $M = \{x \in [a,b]: \underline{g}(x) \leqq c\}$ and $\lambda = \sup M$. Since $\underline{g}(a) \leqq c$, M is not void. The cases $\lambda = a$ and $\lambda = b$ are easy and will be omitted. Let $\lambda \neq a$ and $\lambda \neq b$. Then we have $\underline{g}(\lambda) \leqq \underline{\lim}_{x\to\lambda}\underline{g}(x) \leqq c$. If $\overline{g}(\lambda) \geqq d$, then $G(\lambda) \supset A$ and the theorem in this case is proved. Let $\overline{g}(\lambda) < d$. We have $\overline{g}(\lambda) \geqq \overline{\lim}_{x\to\lambda}\overline{g}(x)$. Therefore there exists $\delta > 0$ such that $\overline{g}(x) < d$ whenever $|x-\lambda| < \delta$. Let $\xi \in (\lambda,\lambda+\delta) \cap \Delta$. Then $\xi > \lambda$ implies $\xi \,\overline{\in}\, M$ and $\underline{g}(\xi) > c$. From $|\xi-\lambda| < \delta$ we obtain $\overline{g}(\xi) < d$ and consequently $G(\xi) \supset A$, which proves the theorem.

Corollary 2. *If $G \in F_\Delta$, $\Delta = [a,b] \subset R$ and $\lambda \in G(a) \vee G(b)$, then there exists $\xi \in [a,b]$ such that $\lambda \in G(\xi)$.*

Corollary 3. *If $G \in F_\Delta$, $\Delta = [a,b] \subset R$, then the union of the values of G is a compact interval.*

4. Interval functions of a real variable. Differentiation.

In the previous chapter we assumed that the interval functions are defined in some metric space L. In the special case when L = R we speak of interval functions of a real variable.

Derivatives and S-derivatives. Let the interval function G be defined on a neighbourhood of $x_o \in R$. Then the interval $G'(x_o) = S\text{-}\lim_{h\to o}(G(x_o+h)-G(x_o))/h$ is called the S-derivative of G at the point x_o. The interval $G'(x_o)$ (finite or infinite as the case may be) always exists. If the limit $\lim_{h\to o}(G(x_o+h)-G(x_o))/h$ exists and is a finite interval, we say that G is differentiable at x_o and that $G'(x_o) = \lim_{h\to o}(G(x_o+h)-G(x_o))/h$ is the derivative of G at x_o.

It is immediately seen that the S-derivative is a generalization of the concept of derivative both of interval and real function. Therefore the utilization of identical notations does not lead to confusion.

The definition of the S-derivative may be formulated also in the following form: The interval D is an S-derivative of G at x_o if D is the interval of minimum width such that $G(x_o+h)-G(x_o) \subset hD+O(h)$, where O(h) is an interval function with the property O(h)/h → 0 when h → 0.

Note that every real function has a S-derivative, and the S-derivative of a real function can be an interval of nonzero width. For example, the S-derivative of f(x) = |x| at x = 0 is the interval [-1,1].

Theorems for differentiation.

Theorem 13. *Let* $G(x) = [\underline{g}(x),\overline{g}(x)]$ *be defined on some interval* $\Delta \subset R$. *Then* $G'(x) = \underline{g}'(x) \vee \overline{g}'(x)$.

Proof. $G'(x) = \text{S-lim}_{h\to 0}(G(x+h)-G(x))/h =$

$\text{S-lim}_{h\to 0}[(\underline{g}(x+h)-\underline{g}(x))/h \vee (\overline{g}(x+h)-\overline{g}(x))/h] =$

$(\text{S-lim}_{h\to 0}(\underline{g}(x+h)-\underline{g}(x))/h) \vee (\text{S-lim}_{h\to 0}(\overline{g}(x+h)-\overline{g}(x))/h) =$

$\underline{g}'(x) \vee \overline{g}'(x)$.

Definition. *We shall say, that the interval function G is w-increasing in the interval* Δ, *if* $x_1 < x_2$, $x_{1,2} \in \Delta$ *implies* $G(x_1) \subset G(x_2)$; *G is w-decreasing in* Δ, *if* $x_1 < x_2$, $x_{1,2} \in \Delta$ *implies* $G(x_1) \supset G(x_2)$.

Definition. *Assume that F and G are two w-monotonic interval functions defined on some interval* Δ. *We shall say that F and G are equally w-monotonic on* Δ *(e. w-m. on* Δ*), if either both F and G are w-increasing* *on Δ or both are w-decreasing on Δ. We say that F and G are differently w-monotonic on Δ (d. w-m. on Δ), if one of the functions F, G is w-increasing on Δ and the other is w-decreasing on Δ.*

Theorem 14. *Let F and G be defined on $\Delta \subset R$. Then $F'(x) \oplus G'(x) \subset (F(x)+G(x))' \subset F'(x) + G'(x)$ and moreover*

a) If F and G are e. w-m. on Δ and at least one of them is differentiable, then $(F(x)+G(x))' = F'(x) + G'(x)$;

b) If F and G are d. w-m. on Δ, F and F + G are e. w-m. on Δ and G is differentiable, then $(F(x) + G(x))' = F'(x) \oplus G'(x)$;

c) If F and G are d. w-m. and both F, G are differentiable on Δ, then $(F(x) + G(x))' = F'(x) \oplus G'(x)$.

Proof. From section 1, property 13 we have $(F(x+h)-F(x))/h \oplus (G(x+h)-G(x))/h \subset ((F(x+h)+G(x+h))-(F(x)+G(x)))/h \subset (F(x+h)-F(x))/h + (G(x+h)-G(x))/h$. Hence for $h \to 0$ Theorem 2 implies the formulated inclusions.

In the case a) we have $((F(x+h)+G(x+h))-(F(x)+G(x)))/h = (F(x+h)-F(x))/h + (G(x+h)-G(x))/h$. Setting $h \to 0$ and using Theorem 3 we obtain the equation.

In the case b) we have $((F(x+h)+G(x+h)) - (F(x)+G(x)))/h = (F(x+h)-F(x))/h \oplus (G(x+h)-G(x))/h$. Setting $h \to 0$ and using Theorem 3 we obtain the necessary equation.

Case c) is trivial.

Continuous analogous of Theorem 14 can be proved for other operations in $I(\bar{R})$ as well.

<u>*Theorem 15*</u>. *Let φ be a continuous real function, defined on the interval Δ, $\varphi(t) \in \Delta_1$ for $t \in \Delta$, F be an interval function defined on Δ_1. $H(t) = F(\varphi(t))$ is an interval function, defined on Δ. Assume also that φ is differentiable at $t_o \in \Delta$. Then we have $H'(t_o) \subset F'(\varphi(t_o))\cdot\varphi'(t_o)$. Moreover, if $\varphi'(t_o) \neq 0$ then $H'(t_o) = F'(\varphi(t_o))\varphi'(t_o)$.*

<u>Proof</u>. We have $(H(t_o+h)-H(t_o))/h = (F(\varphi(t_o+h))-F(\varphi(t_o)))/h = ((F(\varphi(t_o+h))-F(\varphi(t_o)))/(\varphi(t_o+h)-\varphi(t_o)))\cdot((\varphi(t_o+h) - \varphi(t_o))/h)$. Since $\varphi(t_o+h) \to \varphi(t_o)$ for $h \to 0$ and using Theorem 4 we obtain $H'(t_o) = S\text{-}\lim_{h\to o}(F(\varphi(t_o+h)) - F(\varphi(t_o)))/h \subset S\text{-}\lim_{h\to o}(F(\varphi(t_o+h))-F(\varphi(t_o)))/(\varphi(t_o+h)-\varphi(t_o))\cdot \lim_{h\to o}(\varphi(t_o+h)-\varphi(t_o))/h \subset F'(\varphi(t_o))\cdot\varphi'(t_o)$.

If $\varphi'(t_o) \neq 0$, then φ does not have an extremum at t_o. From the continuity of φ it follows that all numbers in every sufficiently small neighbourhood of $\varphi(t_o)$ are functional values of $\varphi(t)$ when t changes in some neighbourhood of t_o. Then $S\text{-}\lim_{h\to o}(F(\varphi(t_o+h))-F(\varphi(t_o)))/(\varphi(t_o+h)-\varphi(t_o)) = S\text{-}\lim_{u\to o}(F(\varphi(t_o)+u)-F(\varphi(t_o)))/u = F'(\varphi(t_o))$. From the differentiability of φ, $\varphi'(t_o) \neq 0$ and Theorem 4 it follows that $H'(t_o) = S\text{-}\lim_{h\to o}(H(t_o+h)-H(t_o))/h = S\text{-}\lim_{h\to o}(F(\varphi(t_o+h))-F(\varphi(t_o)))/(\varphi(t_o+h)-\varphi(t_o))\cdot\lim_{h\to o}(\varphi(t_o+h)-\varphi(t_o))/h = F'(\varphi(t_o))\varphi'(t_o)$.

Mean value theorems. In what follows we shall consider some theorems analogous to mean value theorems for the real functions, demonstrating thereby the properties of the S-derivative of an interval function.

Theorem 16. Let F be an interval function, defined on the interval $\Delta = [a,b]$. Then $(F(b)-F(a))/(b-a) \subset \vee_{x\in\Delta}F'(x)$.

Proof. We choose an arbitrary $\varepsilon > 0$ and denote $E = [-\varepsilon,\varepsilon]$. Since $F'(x) = S\text{-}\lim_{h\to o}(F(x+h)-F(x))/h$, there exists $\delta_x > 0$, such that $F'(x)+E \supset (F(x+h)-F(x))/h$ for $|h| \leqq \delta_x$. We denote $\Delta_x = (x-\delta_x, x+\delta_x)$. Since Δ is a compact interval and $\cup_{x\in\Delta}\Delta_x \supset \Delta$ there exist $x_1 < x_2 < \ldots < x_n \in \Delta$, such that $\cup_{i=1}^{\infty}\Delta_{x_i} \supset \Delta$. Without loss of generality we may assume that $x_1 = a$, $x_n = b$.

Let $y_i \in \Delta_{x_i} \cap \Delta_{x_{i+1}}$. Then $y_i - x_i < \delta_{x_i}$, $x_{i+1} - y_i < \delta_{x_{i+1}}$ and $F(x_{i+1}) - F(y_i) \subset (F'(x_{i+1})+E)(x_{i+1}-y_i)$, $F(y_i)-F(x_i) \subset (F'(x_i)+E)(y_i-x_i)$ and therefore $\underline{f}(x_{i+1}) - \underline{f}(y_i) \in (F'(x_{i+1})+E)(x_{i+1}-y_i)$, $\underline{f}(y_i) - \underline{f}(x_i) \in (F'(x_i)+E)(y_i-x_i)$.

Thus we have $\underline{f}(b) - \underline{f}(a) = \sum_{i=1}^{n-1} (\underline{f}(x_{i+1}) - \underline{f}(y_i) + \underline{f}(y_i) - \underline{f}(x_i))$
$\in \sum_{i=1}^{n-1} ((F'(x_{i+1})+E)(x_{i+1}-y_i) + (F'(x_i)+E)(y_i-x_i)) \subset$
$(v_{x\in\Delta} F'(x)+E) \sum_{i=1}^{n-1} (x_{i+1}-y_i+y_i-x_i) = (v_{x\in\Delta} F'(x)+E)(b-a)$.

Analogously we have $\overline{f}(b) - \overline{f}(a) \in (v_{x\in\Delta} F'(x)+E)(b-a)$. Then $F(b)-F(a) = [(\underline{f}(b)-\underline{f}(a)) v (\overline{f}(b)-\overline{f}(a))] \subset (v_{x\in\Delta} F'(x)+E)(b-a)$, which implies $(F(b)-F(a))/(b-a) \subset v_{x\in\Delta} F'(x)+E$. Since $\varepsilon > 0$ is arbitrary and $v_{x\in\Delta} F'(x)$ is a closed interval, then $(F(b)-F(a))/(b-a) \subset v_{x\in\Delta} F'(x)$.

<u>*Theorem 17*</u>. *If $F(x)$ is an interval function, defined and S-continuous on $\Delta = [a,b]$ and $F(a) \asymp F(b)$, then there exists $\xi \in [a,b]$, such that $F'(\xi) \ni 0$. Besides, if F is continuous on $[a,b]$ then ξ can be chosen in the interior of $[a,b]$.*

<u>Proof</u>. Without loss of generality we may assume that $F(a) \subset F(b)$. From Corollary 1 it follows that there exist $\xi_1, \xi_2 \in \Delta = [a,b]$, such that $\underline{f}(\xi_1) \leqq \underline{f}(x)$ and $\overline{f}(\xi_2) \geqq \overline{f}(x)$ for $x \in \Delta$. Obviously we can assume that $\xi_1, \xi_2 \neq a$. Suppose first that $\xi_1 = \xi_2 = b$. Then $F(\xi) \subset F(b)$

for all $\xi \in [a,b]$. This implies that $F'(b) = S\text{-}\lim_{\xi\to b}(F(b)-F(\xi))/(b-\xi) \ni 0$. Let now $\xi_2 < b$. Then $(\overline{f}(\xi_2+h)-\overline{f}(\xi_2))/h \geqq 0$ for $h < 0$ and $(\overline{f}(\xi_2+h)-\overline{f}(\xi_2))/h \leqq 0$ for $h > 0$ which implies that $\overline{f}'(\xi_2) \ni 0$. The case $\xi_1 < b$ is considered analogically.

Let F be continuous. If $\xi_1 < b$ or $\xi_2 < b$, then the theorem follows from above considerations. Let $\xi_1 = \xi_2 = b$. Then two cases arise:

a) There exists $c \in (a,b)$ such that $\underline{f}(b) < \underline{f}(c) \leqq \overline{f}(c) < \overline{f}(b)$. Using the fact that F is continuous we have $\underline{f}(x) < \underline{f}(c) \leqq \overline{f}(c) < \overline{f}(x)$ for $b-x < \delta$. Let $d \in (b-\delta,b)$. Then $F(d) \asymp F(c)$ and from the above it follows that there exists $\xi \in [c,d] \subset (a,b)$ such that $F'(\xi) \ni 0$.

b) For every $c \in (a,b)$ one of the equalities $\underline{f}(c) = \underline{f}(b)$ or $\overline{f}(c) = \overline{f}(b)$ holds true. If $\underline{f}(b) = \overline{f}(b)$, then $F(x) = \underline{f}(x) = \overline{f}(x) = \underline{f}(b) = \overline{f}(b)$ for $x \in \Delta$ and thus $F'(x) = 0$ for $x \in (a,b)$. Let $\underline{f}(b) < \overline{f}(b)$ and let $c \in (a,b)$ be such that $\underline{f}(c) = \underline{f}(b)$. Then for $x \in (c-\delta,c+\delta)$ we have $\underline{f}(x) < \underline{f}(b)+(\overline{f}(b)-\underline{f}(b))/2 = (\underline{f}(b)+\overline{f}(b))/2 < \overline{f}(b)$ and consequently $\underline{f}(x) = \underline{f}(b)$ wherefrom $\underline{f}'(x) = 0$ for $x \in (c-\delta,c+\delta)$ and $F'(c) \ni 0$.

More generally Theorem 17 can be formulated as follows:

Theorem 18. Let F be S-continuous on $\Delta = [a,b]$. If $\tau \in (F(b)-F(a))/(b-a)$ then there exists $\xi \in [a,b]$ such that $\tau \in F'(\xi)$. Moreover, if F is continuous on Δ, then ξ can be an interior point of Δ.

Proof. Consider the function $H(x) = F(x)-\tau(x-a)$. We have $H(b) = F(b)-\tau(b-a)$, $H(a) = F(a)$, $H(b)-H(a) = F(b)-F(a)-\tau(b-a) = (b-a)((F(b)-F(a))/(b-a)) \ni 0$ and consequently $H(b) \asymp H(a)$. From Theorem 17 it follows that there exists $\xi \in [a,b]$ such that $H'(\xi) = F'(\xi)-\tau \ni 0$, that is $F'(\xi) \ni \tau$. If F is continuous on Δ then H is continuous on Δ and therefore ξ may be an interior point of Δ.

The S-derivative as an interval function is not S-continuous in general. However, as the following theorem shows, it has some properties which are characteristic for S-continuous functions.

Theorem 19. Let F be a S-continuous function on $\Delta = [a,b]$. If $\tau \in F'(a) \vee F'(b)$ then there exists $\xi \in [a,b]$ such that $\tau \in F'(\xi)$.

Proof. If $\tau \in F'(a)$ or $\tau \in F'(b)$ then the theorem is obvious. Let $\tau \overline{\in} F'(a)$, $\tau \overline{\in} F'(b)$. Without loss of generality we may assume that $F'(a) < \tau < F'(b)$. We shall consider two cases:

a) Let $(F(b)-F(a))/(b-a) \geqq \tau$. Since $F'(a) < \tau$ there exists $c \in (a,b)$ such that $(F(c)-F(a))/(c-a) < \tau$. Consider $H(x) = F(x)-\tau(x-a)$. We have $H(b)-F(a) = (b-a)((F(b)-F(a))/(b-a)-\tau) \geqq 0$ and consequently $H(b) \geqq F(a)$. We also have $H(c)-F(a) = (c-a)((F(c)-F(a))/(c-a)-\tau) < 0$ and consequently $H(c) \leqq F(a)$. We thus have $F(a) \subset H(c) \vee H(b)$. From Theorem 12 it follows that there exists $d \in [c,b]$ such that $H(d) \asymp F(a)$ that is $H(d)-F(a) = F(d)-F(a)-\tau(d-a) \ni 0$ and therefore $\tau \in (F(d)-F(a))/(d-a)$. Using this and Theorem 18 we obtain that there exists $\xi \in [a,d]$ such that $F'(\xi) \ni \tau$.

b) Let now $(F(b)-F(a))/(b-a) < \tau$. Since $F'(b) > \tau$ there exists $c \in (a,b)$ such that $(F(b)-F(c))/(b-c) > \tau$. Further the arguments are as in previous case, with the only difference that the places of a and b are changed.

Corollary 4. *If F is S-continuous on $\Delta = [a,b]$ then $F'(x)$ has the property $F'(x) \cap S\text{-}lim_{t\to x}F'(t) \neq \emptyset$ for every $x \in \Delta$.*

Corollary 5. *If F is S-continuous on $[a,b]$ and $\underline{f}$, $\overline{f}$ are differentiable at $x_o \in [a,b]$, then $F'(x_o) \subset S\text{-}lim_{x\to x_o} F'(x)$.*

This corollary is true not only for differentiable but for continuous functions as well. More precisely the following theorem holds true.

<u>*Theorem 20*</u>. *If $F(x) = [f_1(x), f_2(x)]$ is continuous on $\Delta = [a,b]$ then $F'(x_o) \subset S\text{-}lim_{x\to x_o} F'(x)$ for every $x_o \in \Delta$.*

<u>Proof</u>. Let $x_o \in \Delta$. If $F'(x_o)$ is a number (degenerate interval) then f_1 and f_2 are differentiable and the theorem follows from Corollary 5. Let $F'(x_o) = [r,s]$ be a nondegenerate interval and let $\lambda \in (r,s)$. Since $F'(x_o) = [r,s] = f_1'(x_o) \vee f_2'(x_o)$ then there exist $i \in \{1,2\}$ and there exists a sequence $x_1, x_2, \ldots, x_n, \ldots,\ x_n \in \Delta$, such that $\lim_{n\to\infty} x_n = x_o$ and $\lim_{n\to\infty}(f_i(x_n)-f_i(x_o))/(x_n-x_o) \in [r,\lambda]$. Thus it follows that for $n > \nu$ it holds that $p_n = (f_i(x_n)-f_i(x_o))/(x_n-x_o) < \lambda$. From Theorem 18 it follows that there exists $\xi_n \in (x_n \vee x_o)$ such that $p_n \in f_i'(\xi_n)$. Since $x_n \to x_o$, we have $\xi_n \to x_o$. Analogically we determine j, q_n and η_n such that $f_j'(\eta_n) \ni q_n > \lambda$. Then $S\text{-}\lim_{x\to x_o} F'(x) \supset S\text{-}\lim_{n\to\infty} F'(\xi_n) \supset S\text{-}\lim_{n\to\infty} f_i'(\xi_n) \supset S\text{-}\lim_{n\to\infty} p_n \leqq \lambda$ and $S\text{-}\lim_{x\to x_o} F'(x) \supset S\text{-}\lim_{n\to\infty} F'(\eta_n) \supset S\text{-}\lim_{n\to\infty} f_j'(\eta_n) \supset S\text{-}\lim_{n\to\infty} q_n \geqq \lambda$. Using that $S\text{-}\lim_{x\to x_o} F'(x)$ is an interval, we obtain that $S\text{-}\lim_{x\to x_o} F'(x) \ni \lambda$ for

every $\lambda \in (r,s)$. Therefore $S\text{-}\lim_{x \to x_o} F'(x) \supset (r,s)$. But since $S\text{-}\lim_{x \to x_o} F'(x)$ is a closed interval, we have $S\text{-}\lim_{x \to x_o} F'(x) \supset [r,s] = F'(x_o)$.

Other theorems on S-derivative. By means of the S-derivative a number of assertions for interval and real functions can be formulated. These assertions do not presume differentiability (in the familiar sense) of the functions invalued.

Using the mean-value theorems given above the following theorem can be easily verified; in its formulation we use the notation $w_F(x) = w(F(x))$.

Theorem 21. Let F be defined on $\Delta = [a,b]$. Then

a) F is monotone increasing on Δ exactly when $F'(x) \geqq 0$ for $x \in \Delta$;

b) F is monotone decreasing on Δ exactly when $F'(x) \leqq 0$ for $x \in \Delta$;

c) F is w-increasing on Δ exactly when $w_F'(x) \geqq 0$ for $x \in \Delta$;

d) F is w-decreasing on Δ exactly when $w_F'(x) \leqq 0$ for $x \in \Delta$;

e) F satisfied a Lipschitz condition on Δ with constant K (in the sense that $\|F(x)-F(y)\| \leqq K|x-y|$ for any $x,y \in \Delta$), exactly when $\|F'(x)\| \leqq K$, $x \in \Delta$.

These assertions apply for real functions as well. For example $f(x) = |x|$ is a Lipschitzian function with a constant 1. We have $f'(x) = \{-1,\ x<0;\ [-1,1],\ x=0;\ 1,\ x>0\}$ which implies $\|f'(x)\| \leq 1$.

Hausdorff distance. Differentiation of sequences of interval functions. Let F and G be interval functions S-continuous on a closed sub-set Ω of the metric space L. By Hausdorff distance $r(F,G)$ between the functions F and G we shall mean the Hausdorff distance between the graphs of F and G considered as closed point sub-sets of $L \times R$. By Hausdorff distance between arbitrary interval functions we mean the Hausdorff distance between their complete graphs.

Let $\phi = (z_\phi, y_\phi) \in L \times R$. By $\phi \in F$ we shall denote that the point ϕ belongs to the graph of F, that is $y_\phi \in F(z_\phi)$. Then the Hausdorff distance between two functions can be written as follows: $r(F,G) = \max\{\sup_{\phi\in F}\inf_{\Gamma\in G}\rho(\phi,\Gamma),\ \sup_{\Gamma\in G}\inf_{\phi\in F}\rho(\phi,\Gamma)\}$, where ρ is the metric in $L \times R$. If ρ_1 is the metric in L then ρ can be for example $\rho(\phi,\Gamma) = \max\{\rho_1(z_\phi, z_\Gamma), |y_\phi - y_\Gamma|\}$. The number $h(F,G) = \sup_{\phi\in F}\inf_{\Gamma\in G}\rho(\phi,\Gamma)$ is called one-sided Hausdorff distance from F to G. Obviously $r(F,G) = \max\{h(F,G), h(G,F)\}$. We shall use a simple lemma for estimation of the Hausdorff distance.

Lemma 1. *If for every $z \in \Omega$ and $y \in F(z)$ there exist $\xi \in \Omega$ and $\eta \in G(\xi)$, such that $\rho_1(z,\xi) \leqq \alpha$, $|y-\eta| \leqq \alpha$, then we have $h(F,G) \leqq \alpha$. Conversely, if $h(F,G) < \alpha$ then for every $z \in \Omega$ and $y \in F(z)$ there exist $\xi \in \Omega$ and $\eta \in G(\xi)$ such that $\rho_1(z,\xi) < \alpha$ and $|y-\eta| < \alpha$.*

Proof. Assume that for every $z \in \Omega$ and $y \in F(z)$ there exist $\xi \in \Omega$ and $\eta \in G(\xi)$ such that $\rho_1(z,\xi) \leqq \alpha$ and $|y-\eta| \leqq \alpha$. Let $\phi \in F$, $\phi = (z,y)$. Then there exist $\xi \in \Omega$ and $\eta \in G(\xi)$ such that $\rho_1(z,\xi) \leqq \alpha$ and $|y-\eta| \leqq \alpha$ that is $\overline{\Gamma} = (\xi,\eta) \in G$ and $\rho(\phi,\overline{\Gamma}) \leqq \alpha$. Therefore $\inf_{\Gamma\in G}\rho(\phi,\Gamma) \leqq \rho(\phi,\overline{\Gamma}) \leqq \alpha$ for every point $\phi \in F$ which implies $h(F,G) = \sup_{\phi\in F}\inf_{\Gamma\in G}\rho(\phi,\Gamma) \leqq \alpha$.

The second part of the theorem is proved analogously.

Theorem 22. *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of interval functions defined on $\Delta = [a,b]$ which converges to the interval function F. If H is S-continuous on Δ and such that $h(F_n',H) \to 0$ for $n \to \infty$ then $F'(x) \subset H(x)$ for $x \in \Delta$.*

Proof. Let $\overline{x} \in \Delta$ and $\varepsilon > 0$ fixed. From the S-continuity of H it follows that there exists $\delta > 0$ such that $H(\overline{x})+E \supset H(x)$ for $|x-\overline{x}| < \delta$ where $E = [-\varepsilon,\varepsilon]$. Since $h(F_n',H) \to 0$ then for $n > \nu$ we have $h(F_n',H) < \min\{\delta/2,\varepsilon\}$.

Let $t \in \Delta$, $|t-\overline{x}| < \delta/2$. From Theorem 16 we have $(F_n(t)-F_n(\overline{x}))/(t-\overline{x}) \subset \vee_{p \in [t \vee \overline{x}]} F_n'(p)$. Let $p \in [t \vee \overline{x}]$ and $y \in F_n'(p)$. From Lemma 1 it follows that there exist $x_p \in \Delta$ and $z \in H(x_p)$ such that $|p-x_p| < \min\{\delta/2, \varepsilon\}$ and $|y-z| < \min\{\delta/2, \varepsilon\}$ whenever $|x_p-\overline{x}| \leqq |x_p-p|+|p-\overline{x}| \leqq \delta/2 + \delta/2 = \delta$ and $H(\overline{x})+2E \supset H(x_p)+E \supset y+E \ni z$. Therefore $H(\overline{x})+2E \supset \vee_{p \in [t \vee \overline{x}]} F_n'(p) \supset (F_n(t)-F_n(\overline{x}))/(t-\overline{x})$. Thus for $n \to \infty$ we have $H(\overline{x})+2E \supset (F(t)-F(\overline{x}))/(t-\overline{x})$ and for $t \to \overline{x}$ we obtain $H(\overline{x})+2E \supset F'(\overline{x})$. Since ε is arbitrary fixed and $H(\overline{x})$ is a closed interval then $H(\overline{x}) \supset F'(\overline{x})$ for every $\overline{x} \in \Delta$.

The above theorem is an analogue of the corresponding theorem for differentiation of sequences of real functions. We note that the uniform convergency is replaced by one-sided Hausdorff convergency, which naturally leads to a weaker result (inclusion instead of equality).

Since the application of Theorem 22 demands certain knowledge on Hausdorff distance we give below two lemmas, which will be used further on.

Lemma 2. *Let F, G and H are defined on a closed sub-set Ω of L. If $H(z) \subset F(z)$ for $z \in \Omega$ then $h(H,G) \leqq h(F,G)$.*

The proof follows trivially from Lemma 1.

Lemma 3. Let F be S-continuous on a closed sub-set Ω of R^2 and let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of real functions defined on the interval Δ such that:

i) $(x,\varphi_n(x)) \in \Omega$ for $x \in \Delta$;

ii) $\{\varphi_n\}_{n=1}^{\infty}$ is uniformly convergent on Δ to the Lipschitzian with constant m function φ. Then $h(H_n,H) \to 0$ for $n \to \infty$ where $H_n(x) = F(x,\varphi_n(x))$, $H(x) = F(x,\varphi(x))$.

Proof. Choose $\varepsilon > 0$ and $x \in \Delta$. From the S-continuity of F it follows that there exists $\delta_x < \varepsilon$ such that $F(x,\varphi(x))+E \supset F(t,y)$ whenever $|t-x| < \delta_x$ and $|\varphi(x)-y| < \delta_x$. Denote $\Delta_x = (x-\delta/2,x+\delta/2)$. Since $\cup_{x\in\Delta}\Delta_x \supset \Delta$ and Δ is a compact interval then there exist $x_1,x_2,\ldots,x_n$ such that $\cup_{i=1}^{n}\Delta_{x_i} \supset \Delta$. From the uniform convergence it follows that for $n > \nu$ we have $|\varphi_n(x)-\varphi(x)| < \min_{i=1,\ldots,n}\{\delta_{x_i}/2,\delta_{x_i}/2m\}$. Let $n > \nu$, $\bar{x} \in \Delta$ and $\bar{z} \in F(\bar{x},\varphi_n(\bar{x}))$. There exist x_i such that $\bar{x} \in \Delta_{x_i}$ and therefore $|\bar{x}-x_i| \leqq \delta_{x_i}/2$, $|\varphi_n(\bar{x})-\varphi(x_i)| \leqq |\varphi_n(\bar{x})-\varphi(\bar{x})| + |\varphi(\bar{x})-\varphi(x_i)| \leqq \delta_{x_i}/2+m\delta_{x_i}/(2m) = \delta_{x_i}$ implying $F(x_i,\varphi(x_i))+E \supset F(\bar{x},\varphi(\bar{x})) \ni z$. Therefore there exist $z_i \in F(x_i,\varphi(x_i))$ such that $|\bar{z}-z_i| < \varepsilon$. Thus we have $|\bar{x}-x_i| < \delta_{x_i} < \varepsilon$ and $|\bar{z}-z_i| < \varepsilon$ and Lemma 1 implies $h(H_n,H) < \varepsilon$ for $n > \nu$, that is $h(H_n,H) \to 0$ for $n \to \infty$.

5. Integration of interval functions of a real argument.

Definition. *Let* $F(x) = [\underline{f}(x), \overline{f}(x)]$ *be an interval function defined on the interval* $\Delta = [a,b]$. *If* $\underline{f}$ *and* $\overline{f}$ *are integrable real functions on* Δ *we say that* F *is integrable on* Δ *and write* $\int_a^b F(x)dx = \left[\int_a^b \underline{f}(x)dx, \int_a^b \overline{f}(x)dx\right]$.

Integration of S-continuous functions.

Theorem 23. *Let* F *be an integrable interval function on* $[a,b]$ *and let* $G(x) = \int_a^x F(t)dt$. *Then*

i) G *is w-increasing on* $\Delta = [a,b]$;

ii) G *is differentiable almost everywhere (a.e.) on* Δ *and* $G'(x) = F(x)$ *a.e. on* Δ;

iii) *if* F *is S-continuous at* $x_o \in [a,b]$ *then* $G'(x_o) \subset F(x_o)$.

Proof. Let $F(x) = [\underline{f}(x), \overline{f}(x)]$ and $G(x) = [\underline{g}(x), \overline{g}(x)]$. It follows from the definition that $\underline{g}(x) = \int_a^x \underline{f}(t)dt$ and $\overline{g}(x) = \int_a^x \overline{f}(t)dt$. Then $w(G(x+h)) - w(G(x)) = \overline{g}(x+h) - \underline{g}(x+h) - \overline{g}(x) + \underline{g}(x) = \int_x^{x+h} (\overline{f}(t) - \underline{f}(t))dt = \int_x^{x+h} w(F(t))dt \geq 0$. Thereby i) is verified.

It is known from the theory of real functions that $\underline{g}$, $\overline{g}$ are differentiable a.e. on Δ as well (see for instance [8], p. 371). Hence it follows that G is differentiable a.e. on Δ and $G'(x) = \underline{g}'(x) \vee \overline{g}'(x) = \underline{f}(x) \vee \overline{f}(x)$ $= F(x)$ a.e. on Δ.

Let us prove now part iii). In this part the derivative should be considered as S-derivative. We have

$G'(x_o) = \text{S-lim}_{h\to o}(G(x+h)-G(x))/h = \text{S-lim}_{h\to o}(\int_x^{x+h} F(t)dt)/h \subset$ $\text{S-lim}_{h\to o}\vee_{p\in[x_o-h,x_o+h]}F(p) \subset F(x_o)$. Note that the last inclusion is of interest only for the points where G is not differentiable.

Corollary 6. If F is S-continuous on $[a,b]$ *then* $F(G',x) \subset F(x)$ *for* $x \in [a,b]$.

For real functions (under certain assumptions; see for instance [8], p. 378) the equality $f(b)-f(a) = \int_a^b f'(t)dt$ holds true. In this section we shall formulate an analogue of this formula for interval functions. This analogue is based on a theorem for inclusion which will be given below.

Theorem for inclusion.

Lemma 4. Let f and m be two real functions, defined on $\Delta = [a,b]$ *and m be integrable on* Δ. *If* $f'(x) \leq m(x)$ *for* $x \in \Delta$ *then* $f(b)-f(a) \leq \int_a^b m(t)dt$.

The proof is based on the following results:

Lemma 5. *If f is an absolutely continuous real function on Δ and E is a sub-set of Δ such that $\mu(\Delta \setminus E) = 0$ and f is differentiable on E then $f'(x_o) \subset S\text{-}\lim_{\substack{x \to x_o \\ x \in E}} f'(x)$ where μ is the Lebesque measure on R.*

Proof. Let $x_o \in \Delta$. From the absolute continuity of f it follows that $f(x)-f(x_o) = \int_{x_o}^{x} f'(t)dt = \text{sign}(x-x_o)\cdot$

$\int_{[x \vee x_o] \cap E} f'(t)dt \subset (x-x_o) \vee_{p \in [x \vee x_o] \cap E} f'(p)$ and therefore

$f'(x_o) = S\text{-}\lim_{x \to x_o} (f(x)-f(x_o))/(x-x_o) \subset$

$S\text{-}\lim_{x \to x_o} \vee_{p \in [x \vee x_o] \cap E} f'(p) = S\text{-}\lim_{x \to x_o, x \in E} f'(p)$.

Corollary 7. *If f is absolutely continuous on Δ and $f'(x) \geq c$ a.e. on Δ then $f'(x) \geq c$ for $x \in \Delta$.*

Lemma 6. *Let E be a set with measure zero, $E \subset [a,b]$ and ε be a positive number. There exists a monotone increasing and absolutely continuous function $X(x)$ such that $X'(x) = +\infty$ for $x \in E$ and $X(b)-X(a) < \varepsilon$.*

For the proof see for instance [8] p. 378.

Lemma 7. *Let f be defined on $\Delta = [a,b]$. If $f'(x) \geq 0$ a.e. on Δ and $f'(x) > -\infty$ for $x \in \Delta$, then $f(b) \geq f(a)$.*

Proof. Let E be the sub-set of Δ where $f'(x) \geqq 0$ is violated. Choose $\varepsilon > 0$. According to Lemma 6 there exists a monotone increasing and absolutely continuous on Δ function $X(x)$, such that $X'(x) = +\infty$ for $x \in E$ and $X(b)-X(a) < \varepsilon$. Consider the function $g(x) = f(x)+X(x)$. We have $g'(x) \subset f'(x)+X'(x)$. If $x \overline{\in} E$, then $f'(x)+X'(x) \geqq 0$ and therefore $g'(x) \geqq 0$. If $x \in E$ then we have $X'(x) = +\infty$, $f'(x) > -\infty$ and therefore $f'(x)+X'(x) = +\infty \geqq 0$ which implies $g'(x) \geqq 0$. Thus we have $g'(x) \geqq 0$ on Δ and using Theorem 21 we obtain $g(b) \geqq g(a)$ and therefore $f(b)+X(b) \geqq f(a)+X(a)$ and $f(b)-f(a) \geqq X(a)-X(b) \geqq -\varepsilon$. Since ε is arbitrary, then $f(b)-f(a) \geqq 0$.

Proof of Lemma 4. Consider the functions $g_n(x) = \max\{m(x),-n\}$. Obviously g_n are integrable and we have $g_n(x) \geqq m(x)$, $f_n(x) = \int_a^x g_n(t)dt \underset{n\to\infty}{\to} \int_a^x m(t)dt$. Consider the function $f_n(x)-f(x)$. We have $(f_n(x)-f(x))' \subset f_n'(x) \ominus f'(x) \geqq f_n'(x)-m(x)$. Since $f_n'(x) = g_n(x) \geqq m(x)$ a.e. on Δ then $(f_n(x)-f(x))' \geqq 0$ a.e. on Δ. Besides, from $g_n(x) \geqq -n$ and Corollary 7 it follows that $f_n'(x) \geqq -n$. Therefore $(f_n(x)-f(x))' \geqq -n-m(x) > -\infty$. Then from Lemma 7 it follows that $f_n(b)-f(b) \geqq f_n(a)-f(a) = -f(a)$ so that $f(b)-f(a) \leqq \int_a^b g_n(t)dt$. For $n \to \infty$ we obtain $f(b)-f(a) \leqq \int_a^b m(t)dt$. This proves the lemma.

<u>*Theorem 24*</u>. *Let F and M be interval functions, defined on* $\Delta = [a,b]$, $\|M(x)\| < \infty$ *and M be integrable on* Δ. *If* $F'(x) \subset M(x)$ *for* $x \in \Delta$ *then* $F(b) \subset F(a) + \int_a^b M(x)dx$ *and provided* $w(F(a)) \geq w(\int_a^b M(x)dx)$ *then* $F(b) \supset F(a) \oplus \int_a^b M(x)dx$.

<u>Proof</u>. We have $\overline{f}'(x) \leq \overline{m}(x)$ for $x \in \Delta$ and from Lemma 4 it follows that $\overline{f}(b) \leq \overline{f}(a) + \int_a^b \overline{m}(x)dx$. We also have $-\underline{f}'(x) \leq -\underline{m}(x)$ and from Lemma 4 we obtain $-\underline{f}(b) \leq -\underline{f}(a) - \int_a^b \underline{m}(x)dx$ that is $\underline{f}(b) \geq \underline{f}(a) + \int_a^b \underline{m}(x)dx$. The above two inequalities imply the inclusion $F(b) \subset F(a) + \int_a^b M(x)dx$.

Consider now the function $H(x) = F(-x)$ on the interval $[-b,-a]$. We have $H'(x) \subset -M(-x)$ and consequently $H(-a) \subset H(-b) + \int_{-b}^{-a} (-M(-x))dx$. Therefore $F(a) \subset F(b) + \int_{-b}^{-a} (-M(-x))dx = F(b) \ominus \int_a^b M(x)dx$. Using $w(F(a)) \geq w(\int_a^b M(x)dx)$ it is easy to see that $F(b) \supset F(a) \oplus \int_a^b M(x)dx$.

Sufficient conditions for absolute continuity.

Theorem 25. Let f be a real function, defined on $\Delta = [a,b]$ and f' be the S-derivative of f. The following two are equivalent and any of them implies that f is absolutely continuous:

i) There exists a real function $m(x)$ integrable on Δ, such that $\|f'(x)\| \leq m(x)$;

ii) $\|f'(x)\| < \infty$ for $x \in \Delta$ and f' is an integrable on Δ interval function.

Proof. We shall first prove that i) implies absolute continuity of f. Consider the functions $m_n(x) = \min\{m(x),n\}$ which are integrable on Δ. It is easy to see that $\int_a^b m_n(x)dx \to \int_a^b m(x)dx$ for $n \to \infty$. We choose $\varepsilon > 0$. Let n be so large that $\int_a^b m(x)dx - \int_a^b m_n(x) < \varepsilon/2$. Let $[x_\nu, x_\nu+h_\nu]$, $\nu = 1,2,\ldots,k$, be unintersecting intervals on Δ. Denote $\alpha = (x_1,x_2,\ldots,x_n,h_1,h_2,\ldots,h_n)$ and $E_\alpha = \cup_{\nu=1}^{k}[x_\nu,x_\nu+h_\nu]$.

From Lemma 4 it follows that $\sum_{\nu=1}^{k} |f(x_\nu+h_\nu)-f(x_\nu)| \leq$ $\sum_{\nu=1}^{k} \int_{x_\nu}^{x_\nu+h_\nu} m(x)dx = \int_{E_\alpha} m(x)dx \leq \int_{E_\alpha} m_n(x)dx + \varepsilon/2 \leq$ $n\mu(E_\alpha) + \varepsilon/2$. Thus, if $\mu(E_\alpha) = \sum_{\nu=1}^{k} h_\nu < \delta = \varepsilon/(2n)$ then $\sum_{\nu=1}^{k} |f(x_\nu+h_\nu)-f(x_\nu)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$ which implies absolute continuity of f.

We shall now prove the equivalence of i) and ii). From i) it follows that f is absolutely continuous and therefore f is differentiable a.e. on Δ and f' is integrable. From $\|f'(x)\| \leqq m(x)$ it follows that $\|f'(x)\| < \infty$ on Δ. In order to prove that ii) implies i) it is enough to choose $m(x) = \|f'(x)\|$ which is an integrable real function on Δ.

Integration of the S-derivative.

Theorem 26. Let $F(x)$ be an interval function, defined on $\Delta = [a,b]$ and $F'(x)$ be its S-derivative. If $\|F'(x)\| < \infty$ for $x \in \Delta$ and $F'(x)$ is an integrable on Δ interval function, then

i) $F(b) \subset F(a) + \int_a^b F'(x)dx$ *and if F is w-increasing on Δ then* $F(b) = F(a) + \int_a^b F'(x)dx$;

ii) $F(b) \supset F(a) \oplus \int_a^b F'(x)dx$ *by the additional assumption that* $w(F(a)) \geqq w(\int_a^b F'(x)dx)$. *Moreover, if F is w-decreasing then* $F(b) = F(a) \oplus \int_a^b F'(x)dx$.

Proof. From Theorem 24 putting $M(x) = F'(x)$ we obtain the above two inclusions. Since $\|F'(x)\| < \infty$ on Δ and F' is integrable on Δ it follows that $\|\underline{f}'(x)\| < \infty$ and $\|\overline{f}'(x)\| < \infty$ and $\underline{f}$, $\overline{f}$ are integrable on Δ. Using Theorem 25 we see that

$\underline{f}$, $\overline{f}$ are absolutely continuous and therefore $\underline{f}(b) = \underline{f}(a)+\int_a^b \underline{f}'(x)dx$, $\overline{f}(b) = \overline{f}(a)+\int_a^b \overline{f}'(x)dx$. Here we can assume that $\underline{f}'$, $\overline{f}'$ are defined only at the points at which $\underline{f}$, $\overline{f}$ are differentiable, that is $\underline{f}'$, $\overline{f}'$ are defined a.e. on Δ.

Let F be w-increasing. Then we have $\underline{f}'(x) \leqq \overline{f}'(x)$ and consequently $F(b) = [\underline{f}(b),\overline{f}(b)] = [\underline{f}(a)+\int_a^b \underline{f}'(x)dx,$
$\overline{f}(a)+\int_a^b \overline{f}'(x)dx] = [\underline{f}(a),\overline{f}(a)]+[\int_a^b \underline{f}'(x)dx,\int_a^b \overline{f}'(x)dx] =$
$F(a)+\int_a^b F'(x)dx.$

Analogically, if F is w-decreasing we have $\underline{f}'(x) \geqq \overline{f}'(x)$ and $F(b) = [\underline{f}(b),\overline{f}(b)] = [\underline{f}(a)+\int_a^b \underline{f}'(x)dx,\overline{f}(a)+\int_a^b \overline{f}'(x)dx] =$
$[\underline{f}(a),\overline{f}(a)]\circledast[\int_a^b \overline{f}'(x)dx,\int_a^b \underline{f}'(x)dx] = F(a) \circledast \int_a^b F'(x)dx.$

A property of the S-derivative of an absolutely continuous function. It is proved in [7] that the S-derivative of a Lipschitzian function G has a property $F(E,G';x) = F(\Delta,G';x)$ for an arbitrary set $E \subset \Delta$ such that $\mu(\Delta\setminus E) = 0$. This holds true for the S-derivatives of an absolutely continuous function, too.

Theorem 27. If G is absolutely continuous on Δ then for every set $E \subset \Delta$ such that $\mu(\Delta\setminus E) = 0$ we have $F(E,G';x) = F(\Delta,G';x)$ for $x \in \Delta$.

<u>Proof</u>. The inclusion $F(E,G';x) \subset F(\Delta,G';x)$ is obvious. We shall verify the inverse inclusion. Denote by E_1 the set of all points of Δ at which G is differentiable and let $E_2 = E_1 \cap E$. Obviously $\mu(\Delta\setminus E_2) = 0$. Let $x_o \in \Delta$ and $\varepsilon > 0$. Denote $E = [-\varepsilon,\varepsilon]$. Then $F(E_2,G';x_o)+E \supset G'(x)$ whenever $|x-x_o| < \delta$ and $x \in E_2$. Let x be such that $|x-x_o| < \delta$ and $x \bar{\in} E_2$. From Lemma 5 we have $G'(x) \subset S\text{-}\lim_{\substack{t\to x\\ t\in E_2}} G'(t)$. Therefore $G'(x) \subset \vee_{|p-x|<\delta_1, p\in E_2} G'(p)$ $\subset \vee_{|t-x_o|<\delta, t\in E_2} G'(t)$, where $\delta_1 = \delta-|x-x_o|$. Hence $F(E_2,G';x_o)+E \supset \vee_{|t-x_o|<\delta, t\in E_2} G'(t) \supset \vee_{|x-x_o|<\delta} G'(x)$ and consequently $F(E_2,G';x_o)+E \supset F(\Delta,G';x_o)$. Since ε is arbitrary we have $F(E,G';x_o) \supset F(E_2,G';x_o) \supset F(\Delta,G';x_o)$ which proves the theorem.

6. Interval functions of several variables. Directional S-derivatives and partial S-derivatives.

Directional S-derivatives. Let G be an interval function defined on a sub-set Ω of a normed space L.

Definition. Let $\ell \in L$, $\|\ell\| = 1$ and $z \in \Omega$. The intervals

$$D_1(G;z)\ell = S\text{-}lim_{t\to o}(G(z+t\ell)-G(z))/t$$

$$D_2(G;z)\ell = S\text{-}lim_{h\to o,\, h/\|h\|\to\ell}(G(z+h)-G(z))/\|h\|$$

are called directional S-derivatives from first and second kind, respectively, of G at the point z in the direction ℓ.

We obviously have $D_1(G;z) \subset D_2(G;z)\ell \vee (-D_2(G;z)(-\ell))$.

Example. Consider the function

$$G(x,y) = \begin{cases} x & \text{for } y = x^2, \\ 0 & \text{for } y \neq x^2. \end{cases}$$

Then we have $D_1(G;(0,0))\ell = 0$ for every $\ell \in R^2$ with $\|\ell\| = 1$, whereas

$$D_2(G;(0,0))\ell = \begin{cases} [0,1] & \text{for } \ell = (1,0) \text{ and } \ell = (-1,0), \\ 0 & \text{for } \ell \neq (1,0),\ \ell \neq (-1,0),\ \|\ell\|=1. \end{cases}$$

This example shows that the directional derivatives D_1 and D_2 are different in general. The derivative D_2 is more convenient in certain situations, but generally it is easier to work with D_1. It is easy to see that if

G is defined in R^2 and continuously differentiable with respect both to x and y then D_1 and D_2 coincide.

Theorem 28. *Let* $z_1, z_2 \in \Omega$ *and* $\ell = (z_1 - z_2)/\,\|z_1 - z_2\|$.

Then $(G(z_1)-G(z_2))/\,\|z_1-z_2\| \subset \vee_{0\leqq\alpha\leqq 1} D_1(G;\alpha z_1+(1-\alpha)z_2)\ell$

$\subset \vee_{0\leqq\alpha\leqq 1,\,\xi=\pm 1}\, \xi D_2(G;\alpha z_1+(1-\alpha)z_2)(\xi\ell)$.

Proof. Consider the function $H(t) = G(z_2+t\ell)$. Obviously $H'(t) = D_1(G;z_2+t\ell)\ell$. Then $(G(z_1)-G(z_2))/\,\|z_1-z_2\| =$ $(H(\|z_1-z_2\|) - H(0))/\,\|z_1-z_2\| \subset \vee_{0\leqq\alpha\leqq 1} H'(\alpha\,\|z_1-z_2\|) =$ $\vee_{0\leqq\alpha\leqq 1} D_1(G;\alpha z_1+(1-\alpha)z_2)\ell \subset \vee_{0\leqq\alpha\leqq 1,\,\xi=\pm 1} D_2(G;\alpha z_1+(1-\alpha)z_2)(\xi\ell)$.

Partial S-derivatives. In what follows we shall consider the case $L = R^2$. Assume that G is defined on a neighbourhood Ω of (x,y).

Definition. *The intervals* $G'_x(x,y) = S\text{-}lim_{t\to 0}(G(x+t,y)-G(x,y))/t$ *and* $G'_y(x,y) = S\text{-}lim_{t\to 0}(G(x,y+t)-G(x,y))/t$ *are called partial S-derivatives of G with respect to x and y respectively at the point (x,y).*

It is immediately seen that $G'_x(x,y) = D_1(G;(x,y))\ell$ for $\ell = (1,0)$ and $G'_y(x,y) = D_1(G;(x,y))\ell$ for $\ell = (0,1)$.

Theorem 29. *Let* $\ell = (\alpha,\beta) \in R^2$, $\|\ell\| = 1$. *Then for* $i = 1,2$ *we have*

$$D_i(G;(x,y))\ell \subset \alpha F(G'_x;(x,y)) + \beta G'_y(x,y),$$

where by $F(G'_x;(x,y))$ *we denote the complete graph of* G'_x *on* Ω.

Proof. Assume first that $G(x,y) = g(x,y)$ is a real function and let $h = (h_1,h_2) \in R^2$. We have $(g(x+h_1;y+h_2)-g(x,y))/\|h\| = ((g(x+h_1,y+h_2)-g(x,y+h_2))/|h_1|)\cdot(|h_1|/\|h\|) + ((g(x,y+h_2)-g(x,y))/|h_2|)(|h_2|/\|h\|)$. Let $h \to 0$ and $h/\|h\| \to \ell = (\alpha,\beta)$. Then $h_1/\|h\| \to \alpha$ and $h_2/\|h\| \to \beta$. Using that $((x+h_1,y+h_2)-(x,y+h_2))/|h_1| = (h_1,0)/|h_1| = (\text{sign } h_1)(1,0)$ and Theorem 28 we obtain $(g(x+h_1,y+h_2)-g(x,y+h_2))/|h_1| \subset (\text{sign } h_1)\vee_{0\leqq\varepsilon\leqq 1} D_1(g;x+\varepsilon h_1,y+h_2)\cdot(1,0) \subset (\text{sign } h_1)\vee_{|p-x|\leqq\|h\|,\ |q-y|\leqq\|h\|} g'_x(p,q)$. This implies $S\text{-}\lim_{h\to 0,\, h/\|h\|\to\ell}(g(x+h_1,y+h_2)-g(x,y))/\|h\| \subset$
$S\text{-}\lim_{\|h\|\to 0}\vee_{|p-x|\leqq\|h\|,\ |q-y|\leqq\|h\|} g'_x(p,q)\cdot$
$\lim_{\|h\|\to 0} h_1/\|h\| + S\text{-}\lim_{h_2\to 0}((g(x,y+h_2)-g(x,y))/h_2\cdot$
$\lim_{\|h\|\to 0} h_2/\|h\| \subset \alpha F(g'_x;x,y) + \beta g'_y(x,y)$.

Let now $G(x,y) = [\underline{g}(x,y),\overline{g}(x,y)]$ be an interval function. As in Theorem 13 we see that $D_i(G;(x,y))\ell = D_i(\underline{g};(x,y))\ell \vee D_i(\overline{g};(x,y))\ell$, $i = 1,2$. Then

$D_i(G;(x,y))\ell \subset (\alpha F(\underline{g}'_x;(x,y)+\beta\underline{g}'_y(x,y)) \vee$
$(\alpha F(\overline{g}'_x;(x,y))+\beta\overline{g}'_y(x,y)) \subset \alpha[F(\underline{g}'_x;(x,y)) \vee F(\overline{g}'_x;(x,y))] +$
$\beta[\underline{g}'_y(x,y) \vee \overline{g}'_y(x,y)] = \alpha F(G'_x;(x,y)) + \beta G'_y(x,y)$.

<u>*Corollary 8*</u>. *If $G'_y(x,y)$ is a finite interval and $\ell = (1,0)$ then $D_2(G;(x,y))\ell \subset F(G'_x;(x,y))$.*

<u>*Corollary 9*</u>. *If $F(G'_x;(x,y))$ is a finite interval and $\ell = (0,1)$ then $D_2(G;(x,y))\ell \subset G'_y(x,y) = D_1(G;(x,y))\ell$.*

<u>Remark 1</u>. The places of x and y in Theorem 29 and its corollaries can be changed by a symmetry argument.

<u>Remark 2</u>. The situation when i) $\alpha = 0$ and $F(G'_x;(x,y))$ is an infinite interval or ii) $\beta = 0$ and $G'_y(x,y)$ is an infinite interval are of particular interest, since then $\alpha F(G'_x;(x,y)) = [-\infty,+\infty]$ or $\beta G'_y(x,y) = [-\infty,+\infty]$. The inclusion is obvious but provides little information. Let $\beta = 0$. Then $S\text{-}\lim_{\|h\|\to 0} (g(x+h_1,y+h_2)-g(x,y))/\|h\| =$
$(S\text{-}\lim_{h_2\to +0}(g(x,y+h_2)-g(x,y))/h_2\cdot h_2/\|h\|) \vee$
$(S\text{-}\lim_{h_2\to -0}(g(x,y+h_2)-g(x,y))/h_2\cdot h_2/\|h\|) \vee 0 \subset$
$\lim_{h_2\to +0}(h_2/\|h\|\, g'_{y+}(x,y)) \vee \lim_{h_2\to -0}(h_2/\|h\|\, g'_{y-}(x,y)) \vee 0$
$= 0 \vee \lim_{\varepsilon\to 0} g'_{y+}(x,y) \vee (-g'_{y-}(x,y)) = A.$

Obviously A takes values 0, $[0,+\infty]$, $[-\infty,0]$, $[-\infty,+\infty]$, while $0\cdot g'_y(x,y)$ can be 0 and $[-\infty,\infty]$.

Thus we have $D_2(G;(x,y))(1,0) \subset F(G'_x;(x,y))+A$, where $A = \lim_{\varepsilon\to 0} (\varepsilon G'_{y+}(x,y) \vee (-G'_{y-}(x,y))) \vee 0$.

7. Applications to Cauchy problem for first order differential equations.

Let f be a real function, defined on some neighbourhood $D \subset R^2$ of (x_o,y_o). In this section we consider the problem:

$$\begin{aligned} y' &= f(x,y), \\ y(x_o) &= y_o . \end{aligned} \qquad (1)$$

Using the extended segment analysis, developed in the previous section, we can formulate conditions for existence and uniqueness of solutions of (1), which in certain situations are more convenient than the familiar ones.

Uniqueness of the solution.

Theorem 30. *If* $\widetilde{D_1(f;(x_o,y_o))\ell} \vee D_2(f;(x_o,y_o)) \not\ni 0$ *where* $\widetilde{D_1(f;(x_o,y_o))\ell} = S\text{-}lim_{(x,y)\to(x_o,y_o)} D_1(f;(x,y))\ell$ *and* $\ell = (1,f(x_o,y_o))/\|(1,f(x_o,y_o))\|$, *then there exists* $\delta > 0$ *such that problem (1) has at most one solution in* $[x_o,x_o+\delta]$.

Proof. Assume the opposite. Let $y_1(x)$ and $y_2(x)$ be two solutions of (1) which differ in every interval $[x_o,x_o+\delta]$.

Without loss of generality we can suppose that $y_1(x) \leqq y_2(x)$. Indeed, in an opposite situation we shall consider $\min\{y_1(x), y_2(x)\}$ and $\max\{y_1(x), y_2(x)\}$. For simplicity assume that $x_o = y_o = 0$. We shall consider three cases:

1) There exists $\varepsilon > 0$ such that $f(x, y_i(x)) > f(0,0)$ for $i = 1,2$ and $x \in (0,\varepsilon]$. Let n be an arbitrary natural number such that $1/n < \varepsilon$. We consider the curves:

$$c_1: y = y_1(x), \qquad c_2: y = y_2(x)$$

in the interval $[0,1/n]$. We perform the substitution

$$\begin{cases} x = \ell_1\xi - \ell_2\eta \\ y = \ell_2\xi + \ell_1\eta \end{cases} \tag{2}$$

where $(\ell_1, \ell_2) = \ell$. Then the equations for c_1, c_2 become:

$$\begin{aligned} c_1: \ \Theta_1(\xi,\eta) &= \ell_2\xi+\ell_1\eta-y_1(\ell_1\xi-\ell_2\eta) = 0, \\ c_2: \ \Theta_2(\xi,\eta) &= \ell_2\xi+\ell_1\eta-y_2(\ell_1\xi-\ell_2\eta) = 0. \end{aligned} \tag{3}$$

Since we have $\Theta'_{i\xi}(\xi,\eta) = \ell_2-y'_1(\ell_1\xi-\ell_2\eta)\cdot\ell_1 = \ell_1(\ell_2/\ell_1-y'_i(\ell_1\xi-\ell_2\eta)) = \ell_1(f(0,0)-y'_i(\ell_1\xi-\ell_2\eta)) < 0$ provided $\ell_1\xi-\ell_2\eta \in (0,1/n)$, the theorem for existence of implicit functions implies that we can define in a unique way the differentiable functions $\xi_1(\eta)$ and $\xi_2(\eta)$ such that

$$\theta_i(\xi_i(\eta),\eta) = 0 \ , \quad i = 1,2 \ , \tag{4}$$

whenever η is such that $\ell_1\xi_i(\eta)-\ell_2\eta \in [0,1/n]$. It is not difficult to be shown that there exists $\eta_o > 0$ such that $\ell_1\xi_i(\eta)-\ell_2\eta \in [0,1/n]$ for every $\eta \in [0,\eta_o]$ and that $\xi_i(0) = 0$ for $i = 1,2$.

Thus we obtain the functions $\xi_1(\eta)$ and $\xi_2(\eta)$ defined on $[0,\eta_o]$ which satisfy **(4)**. We shall prove that $\xi_1(\eta) \geqq \xi_2(\eta)$ for $\eta \in [0,\eta_o]$. Let $\eta \in [0,\eta_o]$. We determine

$$x_1 = \ell_1\xi_1(\eta)-\ell_2\eta \ , \quad x_2 = \ell_1\xi_2(\eta)-\ell_2\eta \ . \tag{5}$$

We have $y_1(x_1) = \ell_2\xi_1(\eta)+\ell_1\eta$ and $y_2(x_2) = \ell_2\xi_2(\eta)+\ell_1\eta$. Then

$$\xi_1(\eta) = \ell_1x_1+\ell_2y_1(x_1), \quad \xi_2(\eta) = \ell_1x_2+\ell_2y_2(x_2),$$

$$\eta = -\ell_2x_1+\ell_1y_1(x_1) = -\ell_2x_2+\ell_1y_2(x_2).$$

We have $y_1(x_1)-y_2(x_2) = (\ell_2/\ell_1)(x_1-x_2) = f(0,0)(x_1-x_2)$ and $0 = y_2(x_2)-y_1(x_1)-f(0,0)(x_2-x_1) \geqq y_2(x_2)-y_2(x_1)-f(0,0)(x_2-x_1)$. Assume that $x_1 < x_2$. Then $(y_2(x_2)-y_2(x_1))/(x_2-x_1) \leqq f(0,0)$ and therefore there exists $s \in (x_1,x_2)$ such that $y_2'(s) \leqq f(0,0)$ but $y_2'(s) = f(s,y_2(s)) > f(0,0)$. This contradiction shows that $x_1 \geqq x_2$ which implies $\xi_1(\eta)-\xi_2(\eta) = \ell_1(x_1-x_2)+\ell_2(y_1(x_1)-y_2(x_2)) = (x_1-x_2)(\ell_1^2+\ell_2^2)/\ell_1 \geqq 0$. Therefore $\xi_1(\eta) \geqq \xi_2(\eta)$ for $\eta \in [0,\eta_o]$.

By differentiation of (4) we obtain

$$\xi_i'(\eta) = (1+y_i'(x_i)f(0,0))/(y_i'(x_i)-f(0,0)), \quad i = 1,2.$$

Assume that $\xi_1'(\eta) \leq \xi_2'(\eta)$ for $\eta \in (0,\eta_0)$. Then $(\xi_1(\eta)-\xi_2(\eta))' \leq 0$ and therefore $\xi_1(\eta)-\xi_2(\eta) \leq \xi_1(0)-\xi_2(0) = 0$, but $\xi_1(\eta) \geq \xi_2(\eta)$ which implies $\xi_1(\eta) = \xi_2(\eta)$. But this immediately implies $y_1(x) = y_2(x)$ in some interval $[0,\delta]$ which contradicts to the assumption made in the beginning. This contradiction shows that there exists $\bar{\eta} \in (0,\eta_0)$ such that $\xi_1'(\bar{\eta}) > \xi_2'(\bar{\eta})$. Moreover $\xi_1(\bar{\eta}) \neq \xi_2(\bar{\eta})$. Indeed, if $\xi_1(\bar{\eta}) = \xi_2(\bar{\eta})$ it is easy to see that $\xi_1'(\bar{\eta}) = \xi_2'(\bar{\eta})$.

Let $x_1 = \ell_1\xi_1(\bar{\eta})-\ell_2\bar{\eta}$, $x_2 = \ell_1\xi_2(\bar{\eta})-\ell_2\bar{\eta}$. Then we have

$$\frac{1+y_1'(x_1)f(0,0)}{y_1'(x_1)-f(0,0)} = \xi_1'(\bar{\eta}) > \xi_2'(\bar{\eta}) = \frac{1+y_2'(x_2)f(0,0)}{y_2'(x_2)-f(0,0)} \quad ;$$

$$y_2'(x_2)-f(0,0)+y_1'(x_1)y_2'(x_2)f(0,0)-y_1'(x_1)f^2(0,0) >$$
$$y_1'(x_1)-f(0,0)+y_1'(x_1)y_2'(x_2)f(0,0)-y_2'(x_2)f^2(0,0);$$

$$(y'_2(x_2)-y'_1(x_1)(1+f^2(0,0)) > 0;$$

which implies $y_2'(x_2) > y_1'(x_1)$ and therefore $f(x_1,y_1(x_1)) < f(x_2,y_2(x_2))$.

Besides, we have

$$\frac{y_1(x_1)-y_2(x_2)}{x_1 - x_2} = \frac{\ell_2(\xi_1(\bar{\eta})-\xi_2(\bar{\eta}))}{\ell_1(\xi_1(\bar{\eta})-\xi_2(\bar{\eta}))} = \frac{\ell_2}{\ell_1} = f(0,0),$$

and because of $x_1 > x_2$, we have $(x_1-x_2, y_1(x_1)-y_2(x_2)) = k\ell$, where $k = \|x_1-x_2, y_1(x_1)-y_2(x_2)\|$. From Theorem 28 we have

$$0 > \frac{f(x_1,y_1(x_1))-f(x_2,y_2(x_2))}{\|(x_1-x_2,y_1(x_1)-y_2(x_2))\|} \subset \vee_{0\leqq\alpha\leqq 1} D_1(f;(\alpha x_1+(1-\alpha)x_2,$$

$$\alpha y_1(x_1)+(1-\alpha)y_2(x_2))).$$

Then there exist p_n, q_n and r_n such that $0 > r_n \in D_1(f;(p_n,q_n))$, $p_n = \alpha x_1+(1-\alpha)x_2$, $q_n = \alpha y_1(x_1)+(1-\alpha)y_2(x_2)$. Thus $0 < p_n \leqq 1/n$, $0 < q_n \leqq 1/n$ and therefore $p_n \to 0$ and $q_n \to 0$ for $n \to \infty$. Then $\widetilde{D_1(f;(0,0))\ell} = \text{S-lim}_{(x,y)\to(0,0)} D_1(f;(x,y))\ell \supset \text{S-lim}_{n\to\infty} r_n \leqq 0$.

On the other hand, $(y_1(x)-0)/(x-0) \underset{x\to 0}{\to} f(0,0) = \ell_2/\ell_1$ implies

$$D_2(f;(0,0)) = \text{S-lim}_{h=(h_1,h_2)\to 0, h/\|h\|\to\ell} \frac{f(h_1,h_2)-f(0,0)}{\|h\|} \supset$$

$$\text{S-lim}_{x\to 0} \frac{f(x,y_1(x))-f(0,0)}{\|x,y_1(x)\|} \geqq 0.$$

Thus we have $\widetilde{D_1(f;(0,0))\ell} \vee D_2(f;(0,0))\ell \ni 0$. This contradicts to the assumption in the theorem and therefore the theorem follows in this case.

2) There exists $\varepsilon > 0$ such that $f(x,y_i(x)) < f(0,0)$, $i = 1,2$, for $x \in (0,\varepsilon)$ This case is considered analogously to case 1).

3) For every $\varepsilon > 0$ there exist $x_1, x_2 \in (0,\varepsilon)$ and $i,j \in \{1,2\}$ such that $f(x_1,y_i(x_1)) \leq f(0,0) \leq f(x_2,y_j(x_2))$. Let $\varepsilon = 1/n$ and let the above inequalities hold true for $x_1^{(n)}, x_2^{(n)}, i_n, j_n$. Since $x_1^{(n)} \to 0$, $x_2^{(n)} \to 0$, $(y_{i_n}(x_1^{(n)})-0)/(x_1^{(n)}-0) \to f(0,0)$, $(y_{j_n}(x_2^{(n)}-0)/(x_2^{(n)}-0) \to f(0,0)$ for $n \to \infty$ then

$$D_2(f;(0,0))\ell \supset \text{S-lim}_{n\to\infty} \frac{f(x_1^{(n)},y_{i_n}(x_1^{(n)}))-f(0,0)}{\|x_1^{(n)},y_{i_n}(x_1^{(n)})\|} \leq 0,$$

$$D_2(f;(0,0))\ell \supset \text{S-lim}_{n\to\infty} \frac{f(x_2^{(n)},y_{j_n}(x_2^{(n)}))-f(0,0)}{\|x_2^{(n)},y_{j_n}(x_2^{(n)})\|} \geq 0,$$

which implies $D_2(f;(0,0))\ell \ni 0$.

The contradiction obtained proves the theorem.

Theorem 30 is rather inconvenient for application in this formulation. By means of Theorem 29 we can formulate it in the following weaker but more convenient form.

<u>*Theorem 31*</u>. *If $F(f'_x;(x_o,y_o))+f(x_o,y_o)F(f'_y;(x_o,y_o)) \,\overline{\ni}\, 0$, then problem (1) has at most one solution in some interval $[x_o,x_o+\delta]$. Besides, if $f(x_o,y_o) = 0$, it is sufficient for the uniqueness of the solution that $F(f'_x;(x_o,y_o))+A \,\overline{\ni}\, 0$, where $A = 0 \vee \lim_{\varepsilon\to+0} \varepsilon f'_{y+}(x_o,y_o) \vee (-f'_{y-}(x_o,y_o))$.*

Proof. From Theorem 29 we have $D_2(f;(x_o,y_o))\ell \subset \ell_1 F(f_x';(x_o,y_o))+\ell_2 f_y'(x_o,y_o)$ and $D_1(f;(x,y))\ell \subset \ell_1 F(f_x';(x,y))+\ell_2 f_y'(x,y)$ which implies $\widetilde{D_1(f;(x_o,y_o)}\ell \subset \ell_1 F(f_x';(x_o,y_o))+\ell_2 F(f_y';(x_o,y_o))$ and consequently $\widetilde{D_1(f;(x_o,y_o))}\ell \vee D_2(f;(x_o,y_o))\ell \subset \ell_1(F(f_x';(x_o,y_o))+ f(x_o,y_o)F(f_y';(x_o,y_o))$. Since $\ell_1 > 0$ using that $F(f_x';(x_o,y_o))+f(x_o,y_o)F(f_y';(x_o,y_o)) \overline{\ni} 0$ we obtain $\widetilde{D_1(f;(x_o,y_o))}\ell \vee D_2(f;(x_o,y_o))\ell \overline{\ni} 0$ and the uniqueness follows from Theorem 30.

Let $f(x_o,y_o) = 0$. Then $\ell = (1,0)$, $D_1(f;(x,y)) = f_x'(x,y)$ and according to Remark 2 we have $D_2(f;(x_o,y_o)\ell \subset F(f_x';(x_o,y_o))+A$. Since $A \ni 0$ we have $\widetilde{D_1(f;(x_o,y_o))}\ell \subset F(f_x';(x_o,y_o))+A$ which implies $\widetilde{D_1(f;(x_o,y_o))}\ell \vee D_2(f;(x_o,y_o))\ell \subset F(f_x';(x_o,y_o))+A$ and the uniqueness follows from Theorem 30.

Remark 3. We see that the function $f(x,y)$ and its derivatives can be considered only in some sub-set of Ω which contains all solutions. For instance, if $M \leqq f(x,y) \leqq N$ it is sufficient to consider f in the set $\{(x,y)\colon x \geqq x_o, y_o+M(x-x_o) \leqq y \leqq N(x-x_o)\} \cap \Omega$.

Example. Consider the problem $y' = f(x,y) = \sqrt{|y|}+x+1$, $y(0) = 0$. Since $f(x,y) \geqq 0$ for $x \geqq 0$, f can be considered only in the set $\{(x,y): x \geqq 0,\ y \geqq 0\}$. Then $f(x,y) = \sqrt{y}+x+1$, $f(0,0) = 1$, $f'_y(0,0) = f'_{y+}(0,0) = +\infty$, $f'_y(x,y) = 1/(2\sqrt{y})$ $f'_x(x,y) = 1$, so that $F(f'_x;(0,0)) = 1$, $F(f'_y;(0,0)) = +\infty$. Thus we have $F(f'_x;(0,0))+f(0,0)$. $F(f'_y;(0,0)) = +\infty \not\ni 0$, which implies the uniqueness of the solution. It is easy to see that $f(x,y)$ is not Lipschitzian with respect to y.

Example. Consider the problem $y' = f(x,y) = \sqrt{|y|}+x$, $y(0) = 0$. We have $f(0,0) = 0$, $f'_{y+}(0,0) = +\infty$, $f'_{y-}(0,0) = -\infty$, so that $A = 0 \vee \lim_{\varepsilon\to 0}\varepsilon(f'_{y+}(0,0) \vee -f'_{y-}(0,0)) = [0,+\infty]$. We thus obtain $F(f'_x;(0,0))+A = 1+[0,+\infty] = [1,+\infty] \not\ni 0$, which shows that the problem has at most one solution.

Existence of solution of a differential inclusion. Let $G(x,y)$ be an interval function defined on $\Omega = \{x_0 \leqq x \leqq b_1,$ $|y-y_0| \leqq c\}$. The problem

$$y'(x) \subset G(x,y(x)), \quad y(x_0) = y_0, \tag{6}$$

is called a differential inclusion with initial condition. The right-hand side may be an arbitrary multivalued function, but we shall restrict ourself to the situation when it is an interval function. The derivative will be considered in the sense of S-derivative.

Theorem 32. *If G is S-continuous and bounded on Ω interval function, then the problem (6) has a solution in some interval $[x_o,b]$.*

Proof. Let $\|G(x,y)\| \leqq M$ and $b = \min\{b_1, x_o + c/M\}$. We denote $\Delta = [x_o,b]$. Let n be a natural number. We divide Δ into sub-intervals by means of the points $x_o < x_1 < \ldots < x_n = b$, $x_k - x_{k-1} = w(\Delta)/n$ and we form the function $y_n(x) = \{y_o, x=x_o; y_n(x_k)+h_k(x-x_k),\ x_k < x \leqq x_{k+1},$ $k = 0,1,\ldots,n-1\}$ where the number h_k is arbritrary chosen from the interval $F(x_k,y_n(x_k))$. Since $\|y_n'(x)\| \leqq$ $\|v_{x\in\Delta}G(x,y(x))\| \leqq M$ and $|y_n(x)| \leqq y_o + Mw(\Delta)$ we obtain that the functions $\{y_n(x)\}_{n=1}^{\infty}$ are uniformly bounded and equicontinuous. Using the theorem of Arzela-Ascoli we can choose an uniformly convergent sub-sequence. Without loss of generality we may assume that $y_n(x) \to y(x)$ for $n \to \infty$. From the uniform convergency it follows that $y(x)$ is continuous on Δ. We note that Δ is chosen so that $(x,y_n(x)) \in \Omega$ and $(x,y(x)) \in \Omega$ for $x \in \Delta$.

Let us fix $\delta > 0$. The function $F(G,\delta;(x,y)) = [I(G,\delta;(x,y)), S(G,\delta;(x,y))]$ is S-continuous on Ω and from Lemma 3 it follows that $h(H_n,H) \to 0$ for $n \to \infty$, where $H_n(x) = F(G,\delta;(x,y_n(x)))$, $H(x) = F(G,\delta;(x,y(x)))$. Let $n > w(\Delta)/\delta$ that is $x_k - x_{k-1} = w(\Delta)/n < \delta$. Then we have $y_n'(x) \subset F(G,\delta;(x,y_n(x))) = H_n(x)$. From Lemma 2 it follows that $h(y_n',H) \to 0$. Since $y(x)$ is continuous we have that $H(x) = F(G,\delta;(x,y(x))$ is S-continuous on Δ and thus we can apply the theorem for differentiation of sequences of functions (Theorem 22). We obtain $y'(x) \subset H(x) = F(G,\delta;(x,y(x)))$. Noticing that G is S-continuous, then for $\delta \to 0$ we obtain $y'(x) \subset G(x,y(x))$.

Problem (6) can be formulated for interval functions, too, that is we may search an interval function $Y(x)$ such that

$$Y'(x) \subset G(x,Y(x)), \quad Y(x_o) = Y_o,$$

assuming that $Y_o \in I(R)$ and that G is defined on $\Omega \subset R \times I(R)$. It is not difficult to see that this problem has a solution, too, if G is S-continuous and bounded.

<u>Existence of solutions of Cauchy problem for first order differential equations</u>. Let $f(x,y)$ be defined and bounded in some region $\Omega \subset R \times R$, $\Omega = \{x_o \leq x \leq b_1, |y-y_o| \leq c\}$. We look for a solution of the problem

$$y' = f(x,y), \; y(x_o) = y_o. \tag{7}$$

Consider the differential inclusion

$$y'(x) \subset F(x,y(x)), \quad y(x_o) = y_o, \tag{8}$$

where $F(x,y) = F(f;(x,y))$.

<u>*Theorem 33*</u>. *If for some solution $y(x)$ of the differential inclusion (8) defined on $\Delta \ni x_o$ the equality $\int_\Delta w(F(x,y(x)))dx = 0$ holds true then $y(x)$ is a generalized solution of problem (7) in the sense that: $y(x_o) = y_o$ and $y'(x) = f(x,y(x))$ a.e. on Δ.*

<u>Proof</u>. From $\int_\Delta w(F(x,y(x)))dx = 0$ it follows that $F(x,y(x)) \in R$ for almost all $x \in \Delta$ which implies that $y'(x) \in R$ a.e. on Δ that is y is differentiable a.e. on Δ. Since $f(x,y(x)) \subset F(x,y(x))$ we have $f(x,y(x)) = F(x,y(x))$ a.e. on Δ so that $y'(x) \subset f(x,y(x))$ a.e. on Δ and consequently $y'(x) = f(x,y(x))$ a.e. on Δ.

It is more convenient to formulate and use the above theorem in the following form:

<u>*Theorem 34*</u>. *Let $f(x,y)$ be defined on Ω and $|f(x,y)| \leqq k$ for $(x,y) \in \Omega$. Then the problem (7) has a generalized solution if at least one of the following conditions hold true:*

i) For every function φ which is Lipschitzian with a constant k and such that $\varphi(x_o) = y_o$, the equality $\int_\Delta w(F(x,\varphi(x)))dx = 0$ holds true for a suitably chosen interval $\Delta \ni x_o$.

ii) For every continuous function φ such that $\varphi(x_o) = y_o$ and $\varphi'(x) \subset F(f;(x,\varphi(x)))$ for $x \in \Delta_1$, $\Delta_1 \ni x_o$ the *equality* $\int_{\Delta} w(F(x,\varphi(x)))dx = 0$ *holds true for a suitably chosen interval* $\Delta \subset \Delta_1$, $\Delta \ni x_0$.

Theorem 34 gives us a criterion for the existence of a solution of (7). It immediately implies that if f(x,y) is continuous then (7) has a solution. Indeed, if f is continuous then $F(f;(x,y)) = f(x,y)$ and $\int_{\Delta} w(f(x,\varphi(x)))dx = 0$ for every function φ and for every interval Δ. However, this theorem can be applied in more general situations as well.

Example. Let $f(x,y) = \{p, 1/(2K) < y \leq 1/(2K+1);$ $q, 1/(2K+1) < y \leq 1/(2K);\ K = \pm 1, \pm 2, \ldots\}$, $p > 0$, $q > 0$, $p,q \in R$, $x_0 = y_0 = 0$; and let φ be such that $\varphi'(x) \subset F(x,\varphi(x)) = F(f;(x,\varphi(x)))$ on some interval $\Delta \ni x_0$ and $\varphi(0) = 0$. Since $p > o$, $q > 0$ then $\varphi'(x) > 0$ and φ is strictly increasing. Then each of the sets $A_K = \{x \in \Delta: \varphi(x) = 1/K\}$, $K = \pm 1, \pm 2, \ldots$ consists of at most one element. Denote $A = \{x \in \Delta: w(F(x,\varphi(x))) > 0\}$. We have $A \subset \cup_{K=\pm 1}^{\pm\infty} A_K$. The set A is countable and therefore $\mu(A) = 0$. Then $|\int_{\Delta} w(F(x,\varphi(x)))dx| = \int_{A} w(F(x,\varphi(x)))dx \leq \mu(A)\max\{p,q\} = 0$ and consequently $\int_{\Delta} w(F(x,\varphi(x)))dx = 0$ so that Theorem 34 implies that there exists a solution of the problem (7). The function f is not continuous with respect to y in this example and therefore Caratheodory's theorem is not applicable.

<u>Cauchy problem for first order interval differential equations</u>. The Cauchy problem for first order differential equations can be formulated for interval functions as well. Let G(x,Y) be an interval function defined on $\Omega \subset R \times I(R)$, $\Omega \ni (x_o, Y_o)$. We shall look for an interval function Y(x) defined on some interval $\Delta \ni x_o$ such that $(x, Y(x)) \in \Omega$ for $x \in \Delta$ and

$$Y'(x) = G(x, Y(x)) \tag{9}$$

$$Y(x_o) = Y_o. \tag{10}$$

We shall be interested in generalized solutions of this problem in the sence that (9) should be satisfied a.e. on Δ. We also note that in (9) the derivative may be considered as S-derivative.

<u>Definition</u>. A solution Y(x) is called w-maximum on Δ if for any other solution Z(x) of (9), (10) the inclusion $Y(x) \supset Z(x)$ holds true on Δ.

<u>Definition</u>. Let H be an interval function, defined on some sub-set of I(R). We say that H is inclusion isotone if $Y_1 \subset Y_2$ implies $H(Y_1) \subset H(Y_2)$.

<u>Theorem 35</u>. Let G(x,Y) be defined on $\Omega = \{x_o \leqq x \leqq b_1, \; \|Y-Y_o\| < c$ and let it be bounded, S-continuous and inclusion isotone with respect to Y on Ω. Then problem (9), (10) has a unique w-maximum solution

in the interval $\Delta = [x_0, b]$ *where* $b = \min\{b_1, x_0 + c/m\}$ *and* m *is such that* $\|G(x,Y)\| \leqq m$ *for* $(x,Y) \in \Omega$.

The proof of this theorem is given elsewhere (Mathematics and Education in Mathematics, 10^{th} Spring Conference of the Bulg. Math. Soc., 1981, p. 96) and will be omitted.

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